Uncertainty Analysis, Verification and Validation of a Stress Concentration in a Cantilever Beam

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Abstract: Stress concentration is an important effect in structural mechanics produced by a concentrated load or discontinuity in the geometry of a structure; stress concentration factor is the ratio of raised stress and nominal stress. In this paper we compare the results from a bending cantilever beam experiment with the theoretical values and COMSOL finite element simulation. In the experiment a simple cantilever beam with a hole is loaded at the end. This geometric discontinuity causes a stress concentration around the hole and this stress concentration is reported with estimates of the experimental uncertainty. To determine the optimum mesh size for the simulation, a grid convergence study was conducted by monitoring the predicted maximum deflection at the end of the beam and comparing it to the theoretical value. Monte Carlo technique was used to quantify the uncertainties associated with the simulation inputs such as the measured beam dimensions and modulus of elasticity. This paper presents an end-to-end example of uncertainty quantification, model verification and validation.

Keywords: Stress concentration, strain cantilever beam, random and systematic uncertainty.

1. Introduction

In this paper we looked at the stress concentration in a loaded cantilever beam which has a through hole. The Structural Mechanics Module in COMSOL version 4.0a was used to create a model of this beam. Simulations of this loaded beam are compared with theoretical values and experimental results. Also presented are the uncertainties both in simulation and experimentation. The Monte Carlo method was used as the method to estimate the total uncertainty which includes random and systematic contributions.[1]

1.1 Experiment Setup

The material used for the experiment includes a high strength aluminum beam with the hole and the strain gages attached. Measurements are recorded by four strain gages mounted on the beam, three near the hole and one at a location where the nominal stress is equivalent to that of a similar beam without a hole. The strain gage types for the three small sensors next to the hole are called G1388 by the manufacturer (VISHAY measurements group) which have grids no longer than 0.031×0.031 inch, and the forth one is a 125AD strain gage which can be bigger with a gage length of about 0.250 inch. For displaying the strain while loading, we used a digital strain indicator P-3500 made by the same company. The model P-3500 strain indicator is a portable, battery-powered precision instrument for use with resistive strain gages and transducers. The P-3500 will accept full, half, or quarter-bridge inputs[2]. Strain gages are normally connected via a front-panel binding posts. The beam is loaded simply with a hanger and four weights each one pound with a total of four pounds plus the weight of the hanger. Figure 1 shows the set up for the experiment and the strain gages on the beam.
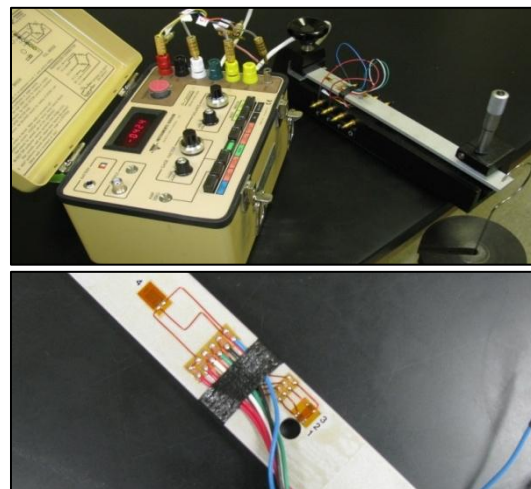

Figure 1. The experiment set up, strain indicator, beam in loading (upper). The beam with the hole and strain gages attached (lower).

1.2 Simulation in COMSOL

The geometry of the beam was created in COMSOL using the CAD device. The beam is made in 3-D space according to the nominal dimensions of the actual beam for the experiment. The fixed end is constrained and the end load is applied as a load per length. Two sets of cut-lines are defined to indicate the position of readings for strain, one through the hole and one at the nominal strain. Figure 1 shows the cut-lines in 3D. In order to find the optimized tetrahedral mesh size for the simulations we conducted grid convergence study based on the stress concentration factor. For this purpose, nine different mesh sizes, from extremely coarse to extremely fine were used and the stress concentration factor is found for each case. Figure 3Figure 1 shows the stress concentration factor versus the number of elements produced in each case.[3]
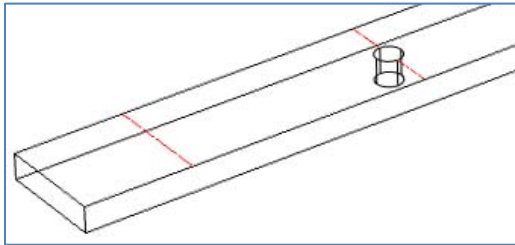

Figure 2. 3D picture of the cut-lines, one through the hole and the other at the location of nominal strain (the location fourth strain gauge in the experiment).
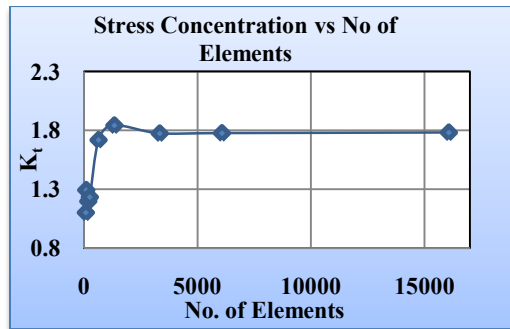

Figure 3. Stress concentration factor versus No. of elements; the finer the mesh size, the higher the No. of elements.

In order to select the mesh size, we looked at the number of elements, average element quality, and solution time. In Table 1, K_t is shown for each mesh case along with the time it took for the simulation to run. According to Figure 4 and

Table 1, the "finer" mesh size seems to be sufficient in that K_t has converged to the second decimal place and the simulation time is not too onerous given the need to run thousands of simulation for the Monte Carlo Methods.

Table 1. Different mesh sizes, stress concentration, number of elements, element quality and solution run time.

Mesh Size	Kt	No of Elements	Average Element Quality	Solution Time (s)
Extremely Coarse	1.100	84	0.2674	1.765
Extra Coarse	1.292	102	0.3396	1.656
Coarser	1.197	191	0.5082	1.563
Coarse	1.232	259	0.5874	1.593
Normal	1.718	644	0.7070	1.734
Fine	1.842	1340	0.7377	1.984
Finer	1.773	3340	0.7746	2.500
Extra Fine	1.776	6082	0.7950	2.860
Extremely Fine	1.781	16080	0.8213	4.422

1.3 Simulation Verification

For verification of our simulation, we compared the maximum deflection of the beam with the theoretical value for the first 500 cases of parametric study. The average value for simulation is equal to 0.142 ± 0.006 inch ($\pm$ one standard deviation) and the average from the theory is equal to 0.141 ± 0.006 inch which is found from Equation 1:

$$\delta = \frac{PL^3}{3EI} \qquad (1)$$

A beam was also modeled once without the hole and the end deflection is found from this simulation to be 0.138 ± 0.006 inch. Both simulations bracket the theoretical value of deflection.

2. Uncertainty in Experiment

There are always errors and uncertainties in the experiments, most common ones stem from the equipment, environment and human repeatability. In order to find the uncertainty in experiment we need to investigate the equipment that we are using. The instruments for this experiment include the strain gages attached to

the beam and the strain indicator P-3500. The accuracy for the gage factor as reported by the manufacture is ±1% and the resolution of the reading from the strain indicator is 1με. Additionally the accuracy is ±0.05% of the reading plus ±3με for the gage factor ≥1.000 in MULT X1 position. (The MULT push button extends the measurement range of the instrument to ±199999με when in the X10 position; the display reading must be multiplied by ten to obtain the correct numerical value. For this test the measurement range of ±19999με is used, so we choose MULT X1).[2] Knowing this information about the equipment gives us the systematic uncertainty; In other words, for systematic uncertainty of the strain indicator, we generate random numbers with normal distribution with mean of zero and standard deviation of 0.0005 (±0.05% of the reading), and mean of zero and standard deviation of 3 (±3με), and a uniform distribution of -0.5 to 0.5 (1με resolution); since all the strain measurements are made with the same device, we can simply consider them correlated which means the random numbers generated for the Monte Carlo samples for all the strain gages are exactly the same. For finding the random errors the experiment was repeated 10 times. The experiment requires the loading the beam with weights connecting each gage sequentially one at-a-time to the quarter Wheatstone bridge strain indicator and reading the initial and final strain; the net value is be calculated by subtracting the initial value from the final value. By having the strain readings for all four strain gages we can find the average and standard deviation for the strain measurement in με.
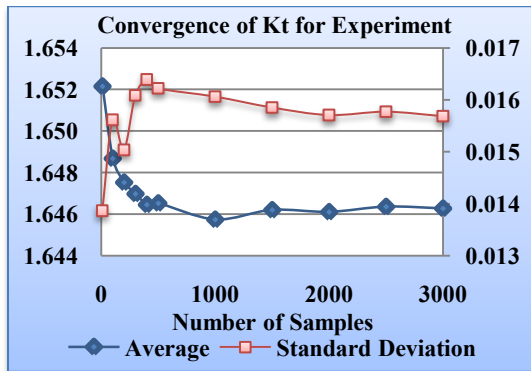

Figure 4. Convergence of average and standard deviation of Kt for COMSOL simulation.

For Monte Carlo simulation of the random error, random numbers are generated with a mean of zero and standard deviation equal to the experimental standard deviation. We run the Monte Carlo for 3000 samples and compute K_t, the concentration factor. Figure 4 shows the convergence of K_t occurs at approximately 1500 samples. Finally the uncertainty of K_t is found by two methods; in the first method, the random and systematic uncertainty for K_t is calculated as standard deviation and then they are combined together with Equation 2 as bellow,

$$u_{K_t} = \sqrt{b_{K_t}{}^2 + S_{K_t}{}^2} \qquad (2)$$

In the second method, the value of K_t is found for all 3000 samples including the random and systematic uncertainty and then the total uncertainty for K_t is calculated by finding the standard deviation.[1] The average value of K_t for both methods is equal to 1.646 and the uncertainty for method (1) and (2) which shows close results is:

$u_{K_t}(1)$=0.0141 (method 1)
$u_{K_t}(2)$=0.0157 (method 2)

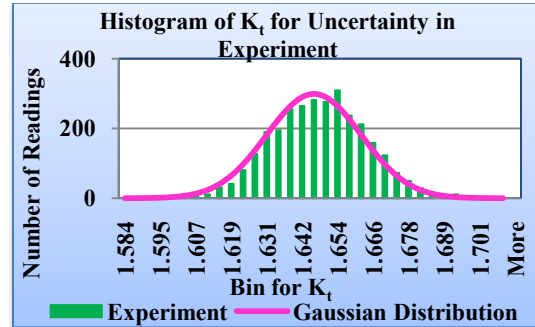In Figure 5 the histogram of K_t for the experiment is plotted.

Figure 5. Histogram of K_t for the experiment

2.1 Stress Distribution around Circular Holes

Discontinuity in geometry results in sudden changes in stress and strain. As we get far enough from the hole the stress distribution will be closer to the nominal value. In order to find the stress distribution we look at the differential form of equilibrium equation,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \quad (3)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \quad (4)$$

Where, X and Y, represent the components of body force. If we solve equations 3 and 4 for two dimensional problems by using the polar coordinate and assuming the body force equal to zero we will have,

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad (5)$$

$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2} \qquad (6)$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \qquad (7)$$

Where, φ is the stress function as a function of r and θ. After applying the boundary condition for a circular hole in a plate subjected to uniform tension of S, the stress at the line across the hole and parallel to the width of the plate is (distributions can be viewed in Figure 6):[4]

$$\tau_{r\theta} = 0 \qquad (8)$$

$$\sigma_\theta = \frac{S}{2}\left(2 + \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right) \qquad (9)$$

Since we are looking at the normal stress distribution on the surface of a cantilever beam, where the shear stress is zero, we can use the derived equations from the uniform loaded plate. Finally, the equation we will use for stress distribution is,

$$\varepsilon_i = A + B[\frac{R}{Z_i}]^2 + C[\frac{R}{Z_i}]^4 \qquad (10)$$

Where, R is the radius of the circular hole and Z_i is the distance of each strain gage to the center of the hole. Table 2 shows the values used in the experiment:

Table 2. Values of R and Z_i

R(in)	Z_1(in)	Z_2(in)	Z_3(in)
0.125	0.145	0.185	0.325

We have three unknowns, A, B, and C in Equation 10 this means three equations are required, which will be provided by installing three stain gages next to the hole. By solving the simultaneous equations we find the constants A, B, and C as bellow in terms of three measured strain,

$$C = 5.86(\varepsilon_1 - \varepsilon_2) - 5.44(\varepsilon_2 - \varepsilon_3) \qquad (11)$$

$$B = 3.49(\varepsilon_1 - \varepsilon_2) - 1.2C \qquad (12)$$

$$A = \varepsilon_1 - 0.743B - 0.552C \qquad (13)$$

At the edge of the hole where $\frac{R}{Z} = 1$ the maximum strain will be:

$$\varepsilon_{max} = A + B + C \qquad (14)$$

The forth strain gage, ε_4, is place where the stress is equivalent to the nominal value of stress if there was not any stress concentration. Since the strain gage factor is different for the three strain gages next to the hole and the forth one, we use $\varepsilon_{4corrected}$ by,

$$\varepsilon_{4corrected} = \varepsilon_4 * \frac{S_{gi}}{S_{g4}} \qquad (15)$$

Where, S_{gi} is the gage factor for the strain gages next to the hole and S_{g4} is the gage factor for the forth one.[5]

3. Uncertainty in Simulation with COMSOL

In order to find how errors propagate through the simulation we performed an uncertainty analysis using the Monte Carlo Method in our COMSOL model. The beam dimensions and load weight are model inputs with measurement uncertainty. Measuring the dimensions of the beam 10 times with the same device gives us the random uncertainty. The accuracy of the caliper used for measuring the dimensions of the beam and the hole, and the accuracy of scale used for measuring the weights are systematic uncertainties. After calculating the total uncertainty we run the Monte Carlo with mean and standard deviation (total uncertainty). For this purpose the parametric sweep function in the COMSOL comes very handy. For this specific model, COMSOL could reasonably handle running about 70 different cases for parametric study; so, to run 3000 simulations in packs of 70 was quite time consuming. It is worth noting that an improvement of COMSOL's ability of running more cases in one simulation would be useful.

3.1 Parametric Sweep

COMSOL has the power of taking variables in parametric format which is a great advantage for optimization and uncertainty studies in which we need to perturb many parameters. Parametric sweep is the key for this type of study. In this paper, we defined the geometry, material properties, and boundary conditions in parametric format to be able to change them as we wish for uncertainty analysis. We used Monte Carlo Method with 3000 samples for our study. So, it is obvious that by having 3000 different

cases it is almost impossible to create a model 3000 times. In order to use parametric study we need to have the values of the first parameter in ascending or descending order. So, we defined a dummy parameter from 1 to 3000 as the first parameter. The rest of the parameters can be in any arbitrary order.

In order to find the random error for each variable, for example for linear dimensions we measured L, w, t, d, and lnom, 10 times with caliper; the average would be the mean value and the standard deviation is used to generate the random numbers. Similar method is used for finding the random values for the weights. But for the systematic error we look at the accuracy of the tool that was used. For example we used a caliper for measuring the linear dimensions which had accuracy of 0.001 inch for small dimensions and a different caliper for bigger dimensions with accuracy of 0.003 inch. It is worth noting that the systematic error is correlated for the dimensions that are measured with the same tool. For Modulus of elasticity, E, we tested the same beam in cantilever loading and we found E from the slope of the linear portion of stress versus strain (Hooke's law) and it was 10e6 psi, but the company that provides the Aluminum beams reports them as high strength aluminum; we expect high strength aluminum beam to have modulus of elasticity about 10.5e6 psi. So, we decided to use 10.4e6 psi as mean and 2.5% error for random uncertainty. The value of systematic error is considered zero for E. The random and systematic errors for each parameter is generated with random number generator in MS Excel and combined to give the total value of the parameters for 3000 cases. The parameters used are showed in Table 3 and mean, random and systematic error for each parameter is tabulated in Table 4.

Table 3. Parameters defined for the parametric sweep.

dummy	Dummy variable in ascending order
L	Length of the beam
w	Width of the beam
t	Thickness of the beam
d	Diameter of the beam
lnom	Distance, nominal stress to fixed end
P	Load
E	Modulus of Elasticity

The number of decimal places for each variable is adjusted based on the instrument used for measuring that variable. For example we used a caliper for measuring the linear dimensions that reads up to three decimal places in inches.

Table 4. The mean, and random and systematic error for each parameter used in COMSOL simulation.

Parameter	Average	Random Error	Systematic Error
L	11.260	0.009	0.003
w	1.002	0.001	0.001
t	0.252	0.002	0.001
d	0.246	0.002	0.001
lnom	1.0205	0.005	0.001
P1	4.010	0.000	0.010
P2	0.064	0.002	0.010
E	1.04E+07	2.5%	0

3.2 Post processing

After running our model for the 3000 Monte Carlo samples we need to process the data. The stress or strain concentration factor is found by dividing the max stress or strain by the nominal stress or strain. By defining a COMSOL max function and a cutline that goes through the center of the hole in the model we can easily get the maximum value of stress or strain. The nominal value of strain is found by using an average function and a cutline that passes through the nominal value of stress and strain. By using the node "Derived Values" and generating "General Evaluation" we get the average and nominal values in tabulated format. By dividing the maximum strain by nominal strain we find K_t for all 3000 cases. The last step is to calculate the average and standard deviation of K_t and report it as stress concentration factor and total uncertainty in simulation. In Figure 6 and Figure 7 you can see the strain distribution through the hole and at the nominal strain location. In these figures, we have plotted the first 50 samples just to illustrate how COMSOL provides graphs for parametric sweep. Figure 8 illustrates the 3 dimensional graph of the stress distribution around the hole.
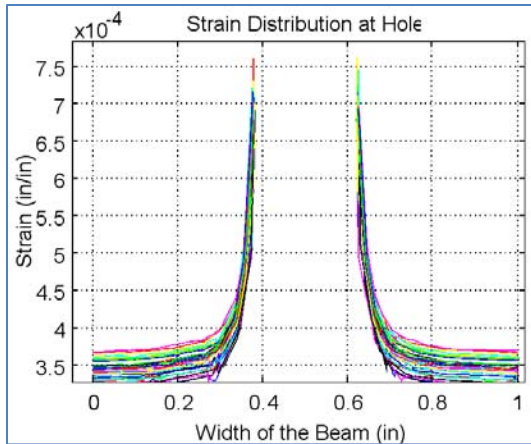
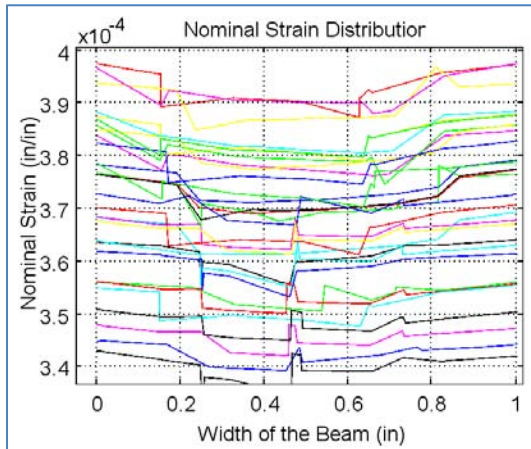**Figure 6.** Strain distribution through the hole.

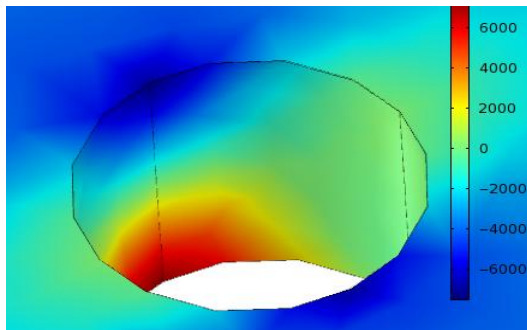**Figure 7.** Strain distribution at nominal strain.

Figure 8. Stress distribution in psi around the hole.

3.3 Convergence of number of samples

Computing power is always a controlling factor in computational simulations. Thus, it is very important to know when the number of samples is enough, or in other words how it convergences. In this study since the model is

relatively simple it may not be a problem if we run 3000 Monte Carlo samples while fewer would be enough, but for a more complicated model it will certainly be essential to know when the number of samples converges. As seen in section 2, the number of samples for experiment converged about 1500. Here, we look at the same graphs for the COMSOL Monte Carlo simulations. In Figure 9 you can see the average and standard deviation of K_t plotted versus number of samples. The total uncertainty for K_t in simulation was found 0.050 (one standard deviation). In Figure 10 the histogram of COMSOL simulated K_t for 3000 samples is compared to Gaussian distribution; note the K_t histogram is not Gaussian. Figure 11 shows histogram of K_t for experiment and simulation in one graph for better comparison.
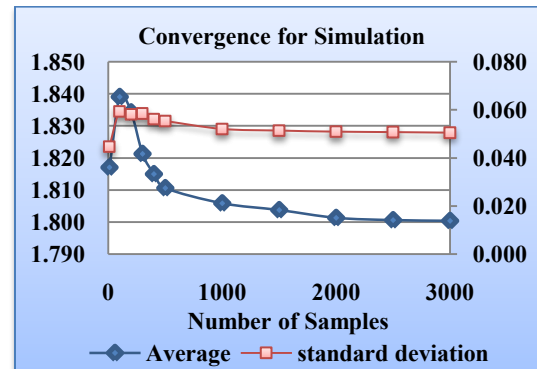

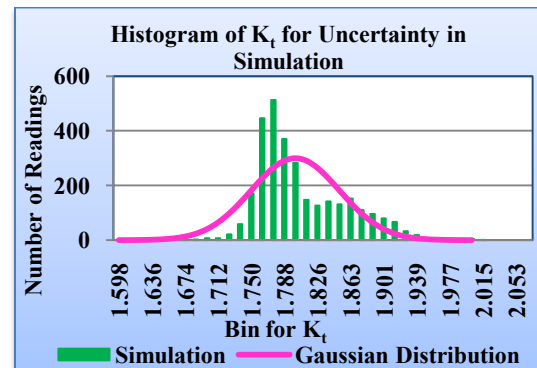**Figure 9.** Convergence of average and standard deviation of Kt for COMSOL simulation.

Figure 10. Histogram of k_t for uncertainty in simulation plotted with comparison to Gaussian distribution.
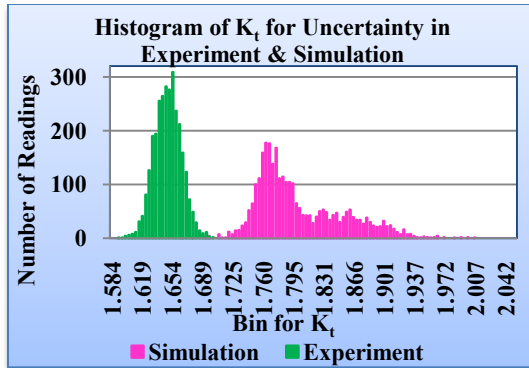
Figure 11. Overlapping histogram of K_t from Experiment and Simulation for better comparison.

4. Results and Conclusions

In this paper, we looked at experimentation, computational simulation and the uncertainty propagation. The stress concentration factor in experiment was found, 1.646 ± 0.0141 and 1.646 ± 0.0157 (from two different methods) and in simulation, 1.80 ± 0.050. Thus we have a comparison error[6] implying there is an un-modeled, un-simulated effect such as the strain gauge sensor glue. Or, perhaps there is an experimental or input uncertainty that is not captured.

Parametric sweep in COMSOL is a simple tool to perform uncertainty analysis with Monte Carlo technique in our computational simulations. Additionally, this same method can be used to learn which parameter has more effect on the final result; in order to prevent consuming time and money on doing different experiments with different uncertainty in parameters, we can use uncertainty analysis in simulation to find out which parameter(s) are controlling factors.

5. References

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6. Appendix

Figures Figure **12** andFigure **13** show the histograms used in for the Monte Carlo Method is the experimental uncertainty analysis and the simulation uncertainty analysis respectively.
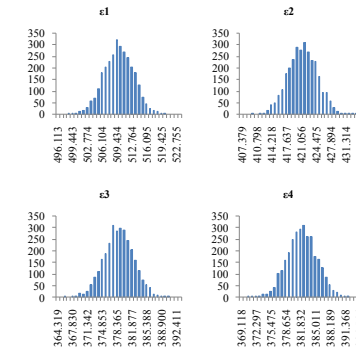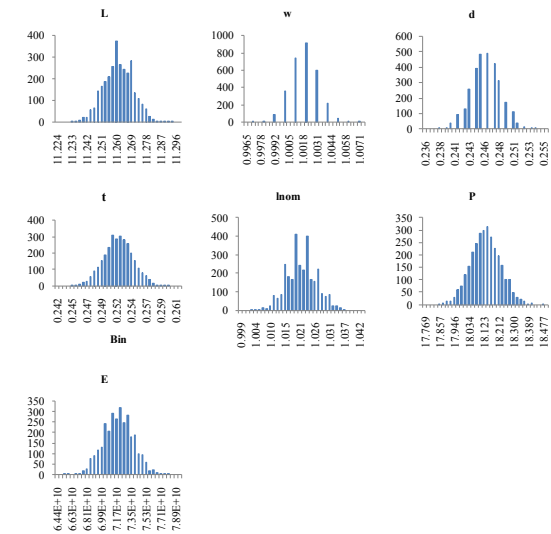

Figure 12. Experimental uncertainty histograms.

Figure 13. Simulation uncertainty histograms.