COMSOL CONFERENCE
2024 BOSTON

Discovery Of Avoided Crossings In Plate Vibrations Using COMSOL

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Abstract

We have recently published the discovery of avoided crossings present in the spectrum of freely vibrating rectangular thin plates [1]. COMSOL Multiphysics was a fundamental tool to accomplish this feat since we easily generated thousands of normal modes by changing the length of one side of the plate and we could classify them into symmetrical and anti-symmetrical modes. It was revealed that for each symmetry sector of the spectrum, avoiding crossings appear. Now we have found that the vibration of a free thin disk sector, also presents avoiding crossings in its spectra and its spacing ratio distribution approximates to the Rosenzweig-Porter model of random matrices theory.

Keywords: Free thin plate vibrations, wave cavities, avoided crossings, Rosenzweig-Porter model, symmetry.

Introduction

The study of the vibrations of thin plates is of great interest in science and engineering, since they appear as structural elements of various systems and devices of great use in society. Using thin plates, bridges or nanoscopic connectors, furniture, machinery and civil constructions are built; also we can even mention the important movement of tectonic plates that can cause earthquakes, affecting the life of large cities. In particular, the understanding of free plate vibrations has been an open problem for more than two hundred years [2] and now, with the development of the aerospace industry, characterization of these vibrations becomes necessary [3], for example it is important to know the resonant effect on the structure of a space satellite in orbit, due to the ignition of an electric motor.

The results of the study on free vibrating thin plates shown in this work confirm once again the von Neumann-Wigner theorem [4, 5], that is, the general spectrum, of the plate studied, shows the existence of a degeneration of levels due to the mixing of all the symmetries of the system, while the repulsion levels or avoided crossings are only observed in each spectrum, corresponding to a single symmetry.

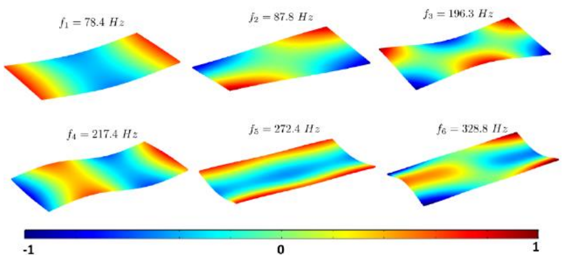

Figure 1.- First six normal modes of the simulation for a thin plate with L = 60 cm. The out-of-plane displacement $W(x, y)$ is displayed using a rainbow scale color in arbitrary units.

Theory

From classical plate theory [2, 6], we can deduce the thin plate equation, considering standing wave solutions of angular frequency ω that cause out-of-plane deformations $W(x, y)$ of a plate of length L along the x axis, with width a on the y axis, and thickness h along the z axis:

$$(\nabla^4 - k^4)W(x, y) = 0. \qquad (1)$$

In this equation the biharmonic operator $\nabla^4 = \nabla^2 \cdot \nabla^2$ is used with the Laplacian operator defined as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and the wave number k given by

$$k^2 = \omega\sqrt{\rho h/D}, \qquad (2)$$

where ρ is the density of the plate; here the flexural stiffness of the body D is defined as

$$D = h^3 E/(12(1 - \nu^2)),$$

with E as the plate´s Young module and ν its Poisson coefficient.

If we consider a thin plate vibrating freely, we have that the boundary conditions for the x axis applied to equation (1) are expressed as

$$\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\Big|_{x=0,L} = 0,$$

$$\frac{\partial^3 W}{\partial x^3} + (2 - \nu)\frac{\partial^3 W}{\partial x \partial y^2}\Big|_{x=0,L} = 0;$$

from these relations, the free boundary conditions for the y axis are obtained, swapping x for y, and changing the length L for the width a of the plate.

Notice that there are no known analytical solutions for equation (1) applying the above free conditions, but solutions for simply supported boundary conditions are exactly known [2, 6]. Theoretically, general solutions to equation (1) with simply supported boundary conditions show that a wave coming from the bulk (interior of the plate) is reflected at the boundary and is redirected back towards the bulk, while in the case of free boundaries, apart from this reflected wave, there is another one, an evanescent wave that travels around the boundary [1].

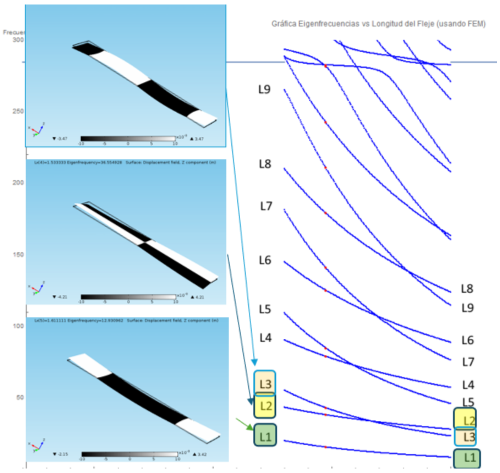

Figure 2.- Left: three out-of-plane modes in black and white that shows the symmetry along and across the free vibrating thin plate. Right: complete spectrum for all the simulated modes, frequency vs length. The three modes belong to different spectral lines.

Simulation

To study equation (1) with free boundary conditions, its flexural spectrum and eigenfunctions (normal modes) must be found numerically. The strategy followed was to simulate the deformations of a rectangular plate in three spatial dimensions, considering that one of its dimensions, its thickness, was very small compared to the two remaining dimensions. A vibrating rectangular aluminum plate was simulated in 3D using the Structural Mechanics module of COMSOL Multiphysics. This finite element method software solves the 3D equations of linear elasticity, also known as Navier-Cauchy equations, applying the variational principle [7].

Using the "Eigenfrequency Study" to find the normal modes of a body and their frequencies, a rectangular block with a width of $a = 344\ mm$, length of $L = 400\ mm$ and thickness $h = 6.35\ mm$ was drawn; with these dimensions we considered that we had a realistic thin rectangular plate, since the thickness was smaller than 10% of the length or width of the body. We choose the typical parameters for aluminum as the block material, that is, $\rho = 2700\ kg/m^3$, $E = 69\ GPa$, and $\nu = 0.33$.

The boundary conditions were of course chosen as free, and after testing good convergence of the eigenfrequencies for different meshes, the "Extremely Fine" mesh was set, corresponding to a cutoff frequency of 967 kHz calculated using equation (2).

To study the behavior of the normal modes of vibration of the plate as the length was varied, a parametric sweep was made for 200 length values, equally spaced, in the range of $L_i = 400\ mm$ to $L_f = 800\ mm$. Determined to carry out laboratory experiments using the Spectral Acoustic Resonance technique, we decided to work in the acoustic regime, considering only simulated modes lower than 20 kHz, so, for each plate length, a search for the first 70 vibration eigenfrequencies $\{f_n\}$, different from zero hertz (rigid body modes), was carried out.

With a normal desktop computer, easily and quickly COMSOL generated information for 14000 normal modes; a sample of these simulated modes are shown in Figure 1, where we can see using a rainbow color scale, the out-of-plane displacement $W(x, y)$ of the first six modes of the plate with a length of $L = 600\ mm$. Notice that in the corners of these modes there is an extreme behavior in amplitude of the displacement, this is evidence of the mixing of evanescent and sinusoidal waves at the edges of the free plate.

To ensure that only out-of-plane modes were analyzed, for each mode, the surface integral of the square of its vertical displacement was calculated in COMSOL and we only considered those modes whose integral is greater than a certain quantity $\varepsilon > 0$; that is, we have an out-of-plane mode if

$$\int_{plate} W^2(x, y)dxdy > \varepsilon. \qquad (3)$$

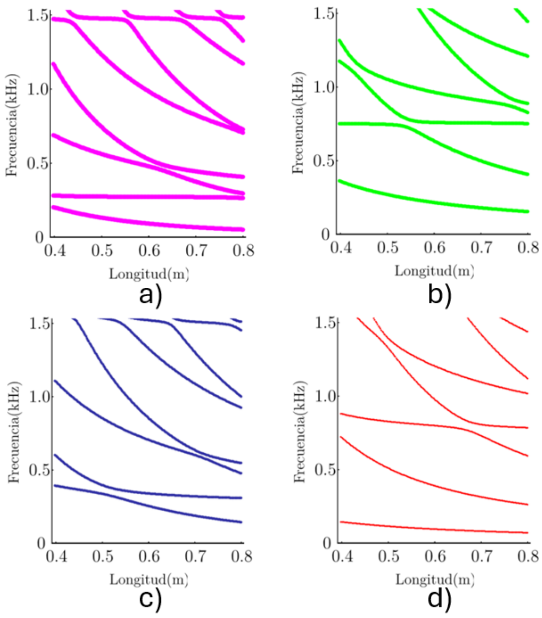

Figure 3.- Portion of the simulated spectrum of a freely vibrating rectangular thin plate, separated by symmetry sectors, using different colors, for out-of-plane modes below 1500 Hz with a) SS, b) SA, c) AS and d) AA symmetry. We can observe only avoided crossings between the spectral lines for each symmetry sector.

Results

To analyze the thousands of generated modes, information on the plate´s length L_j and its corresponding frequencies of the simulated modes $\{f_{j,n}\}$ were extracted from COMSOL; the complete spectrum of the plate was constructed by plotting the frequency of the modes against their length. Fgure 2 shows a portion of the complete spectrum where we can appreciate that there are crossings between the spectral lines.

We were curious to visually pursue how the shape of the mode changes as we moved along a spectral line. At first glance the modes have complex and seemingly wild colorful shapes, so after analyzing a few lines, we perceived that there were certain features that remain unchanged. A crucial point was the decision to change the display of the out-of-plane displacement $W(x, y)$ from color to black and white. As can be seen on the left of Figure 2, the nodal lines of the modes are more evident in black and white, so we immediately realized that the symmetry of the modes was preserved when we moved along a given spectral line.

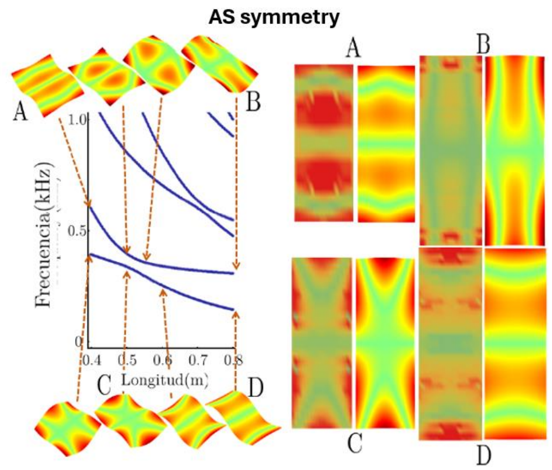

Figure 4.- Left: Avoiding crossings between the spectral lines of the Antisymmetric-Symmetric sector for the out-of-plane modes of the free vibrating rectangular plate. Right: Measured and simulated modes at two spectral lines, before an avoiding crossing, point A with plate length L = 400 mm, *point C with L = 500 mm, and after the avoided crossing, points B and D with plate length L = 800 mm. Notice the swapping of modes´ shapes [5], A to D and C to B.*

Thus, we proceeded to classify the out-of-plane modes according to their symmetry with respect to the x, y axes, that is, we call a mode Symmetric-Symmetric $W^{SS} = W(x, y)$, if its out-of-plane displacement has symmetry along and across the plate. We called a mode Antisymmetric-Symmetric $W^{AS} = W(x, y)$ if it is antisymmetric along the length of the plate and symmetric across its width; similarly, we classify the modes into W^{AA} and W^{SA}. We extracted from COMSOL the information on the displacement $W(x, y)$ corresponding to the corners of the plate and classified each mode according to the sign of these displacements.

By setting $\varepsilon = 0.0005$ and using relation (3) applied to the 14000 simulated modes, and classified according to their symmetry, it was obtained that 28.2% of them were W^{SS} modes, 23.4% W^{SA} modes, 25.5% W^{AS} modes, 22.7% W^{AA} modes and 0.03% modes that could not be classified (the vast majority in-plane modes). Plotting again frequencies vs lengths for each classification, as seen in Figure 3, new spectra were obtained that showed only avoided crossings between the lines for each symmetry sector of the free rectangular thin plate. This behavior was overlooked within the context of wave chaos theory for a system as simple as a rectangular wave cavity [8].

The parameters chosen for the simulation were such that we were able to buy commercial aluminum sheets, cut them and in the laboratory suspend them horizontally with the help of nylon threads. Using the spectral acoustic resonance technique, we could excite and measure the out-of-plane modes of the plates [1]. The right part of Figure 4 shows four pairs of images, where in each pair, the image on the left is the mode measured experimentally and, the image on the right is the mode predicted by the simulation. These image pairs correspond to the simulated modes at spectral points A and C, before an avoided crossing, as well as for points B and D after the avoided crossing, as shown in the left part of Figure 4. It is remarkable that in practical terms, to reconstruct experimentally the complete shape of a mode, it is only necessary to know its symmetry sector, excite the mode, and scan a quarter of the vibrating plate.

To characterize the presence of avoided crossings [9], histograms of the spectral spacing ratios were constructed [1] and it was determined that, the average envelope is closer to a transition model of random matrix theory called Rosenzweig-Porter [10], similar to the distribution shown in Figure 5 b).

Final Remarks

Since this phenomenon is not present for simple supported vibrating plates, our hypothesis is that these avoided crossings are caused by the presence of evanescent waves traveling at the system´s boundary [1]. Evanescent waves are a class of non-Newtonian orbits, whose presence in a system generates spectra that show avoided crossings [11].

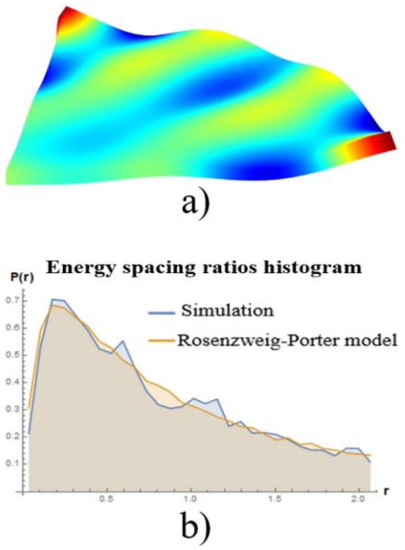

Figure 5.- a) Simulation of a symmetrical normal mode of a disk sector for a 50° angle. b) Histogram of the spectral spacing ratios for 14000 normal modes of a disk sector generated by changing its angle.

It is interesting to investigate whether the phenomenon occurs for other thin plates vibrating freely, plates with spatial symmetry. For the moment, we have been able to corroborate using COMSOL again, that in a freely vibrating disk sector [3], by varying the angle of the sector and obtaining its spectrum, now only classifying the modes into symmetric and antisymmetric ones, avoided crossings are found, whose spectral spacing ratio distribution envelope approximates to the Rosenzweig-Porter model, as seen in Figure 5 b). Also notable in this case, it is the evanescent wave in the border of the plate, manifested through the large amplitudes of the mode at the corners of the disk sector, as shown in Figure 5 a).

Interesting paths are opened to investigate and fully understand the phenomenon of avoided crossings in the spectra of vibrating plates, such as the relationships between the elastic variables and the parameters of the random matrix models, as well as the possible applications generated by applying multiphysics phenomena on the design and development of metamaterials [12, 13].

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Acknowledgements

This work was made possible due to the financial support of CONAHCYT-SNII and UAA Project PIM24-8.