Improved Perfectly Matched Layers for Acoustic Radiation and Scattering Problems

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Abstract: Perfectly matched layers (PML) are an efficient alternative for emulating the Sommerfeld radiation condition in the numerical solution of wave radiation and scattering problems. The key ingredient of the PML formulation is the complex scaling function, which controls the anisotropic damping of the PML. The objective of this study is to propose a modified complex scaling function capable of providing the user with two advantages: (i) minimization of the spurious reflections at the physical domain-PML interface for propagating and evanescent fields at all angles of incidence and (ii) stability with respect to the frequency parameter (which reduces the meshing effort in broadband applications). Numerical results are presented for radiation from a circular piston and for scattering from a rigid sphere. Overall, the modified formulation is more stable at lower frequencies, while some potential difficulties arising in high-frequency radiation problems remain to be addressed.

Keywords: acoustics, radiation, scattering, perfectly matched layers, waves.

1. Introduction

The capability of emulating the Sommerfeld radiation condition, which requires that outgoing waves propagate out towards infinity in the absence of reflecting boundaries, is a critical component of any numerical code concerned with the solution of wave problems. For tools based on bounded computational domains, such as finite-element tools, the finite-sized computational region is truncated by an outer boundary. This boundary represents the ideal interface between the finite region, modeled by the computational domain, and the surrounding infinite medium. In order to satisfy the radiation boundary condition, outgoing waves must traverse such an ideal boundary without being reflected.

One method for defining non-reflecting boundaries consists of surrounding the finite computational region with a perfectly matched layer (PML). The PML is a non-physical layer, inside which the wave equation has been modified with an anisotropic damping, which increases with distance in the direction perpendicular to the interface with the physical domain. The result is that waves entering the PML are absorbed only in the outgoing direction, while the wave components tangential to the interface between the physical domain and the PML remain unaffected. This approach was introduced originally by Bérenger [1] in 1994 for electromagnetic waves. Reviews of PML research, with a particular emphasis on acoustics, can be found for example in References [2] and [3].

The main advantages of the PML formulation, compared to other numerical radiation boundary conditions, are the relatively straightforward implementation via a complex coordinate scaling, and the adaptability to generic convex boundaries [3, 4]. This makes it possible to minimize the size of the computational domain, particularly in those cases where the physical domain cannot be circumscribed easily by spheroids or similar shapes, which are usually required by other numerical radiation boundary conditions like Dirichlet-to-Neumann maps or infinite elements.

One major difficulty associated with PML formulations is the choice of the scaling function, which defines the absorption of the outgoing waves in the PML. In Section 2 it is shown how the scaling function proposed in Ref. [3], and the corresponding versions implemented in Comsol [5], suffer from inaccuracies at very low frequencies ($ka \ll 1$, where k is the acoustic wave number and a is a characteristic size of the computational domain). The source of the inaccuracies lies in the discretization of the evanescent wave field components, which decay steeply inside the PML. Examples are provided for two different test problems: radiation from a baffled rigid circular piston, and plane wave scattering from a rigid sphere.

A modification to the scaling strategies of Refs. [3, 5] is proposed in Sec. 3. The new PML formulation is more accurate at the low to mid frequencies, compared to the scalings of Refs. [3, 5], while the high-frequency performance for radiation problems requires some further work.

2. Inaccuracies at low frequencies, caused by the evanescent wave-field

An analysis of the plane-wave damping in a finite-thickness PML shows that a one-wavelength thick PML, defined as in Refs. [3, 5], reduces spurious reflections to about -100dB relative to the outgoing wave. The continuous analysis also shows that evanescent waves, which are naturally decaying, are absorbed even more strongly, depending on the exponent of the evanescent wave (Fig. 4 in Ref. 3). On the other hand, discretization problems arise with the strongly evanescent wave-field at very low frequencies.

2.1 Radiation from a baffled piston

The first test problem to illustrate the difficulties of discretizing the evanescent wave field at low frequencies is that of the radiation from a baffled rigid circular piston (Fig. 1).

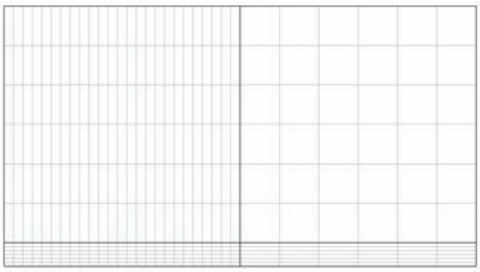

Figure 1. Piston model geometry and mid-resolution mesh with the physical domain in the lower left corner, surrounded by PML domains.

The piston radius is *a*, and the constant acceleration on the piston face is set to a nominal value of c^2/a, with *c* being the speed of sound. The acoustic pressure, sampled at *a*/10 distance from the center of the piston along the axis of symmetry, is plotted as a function of the nondimensional wave number *ka* in Fig. 2.

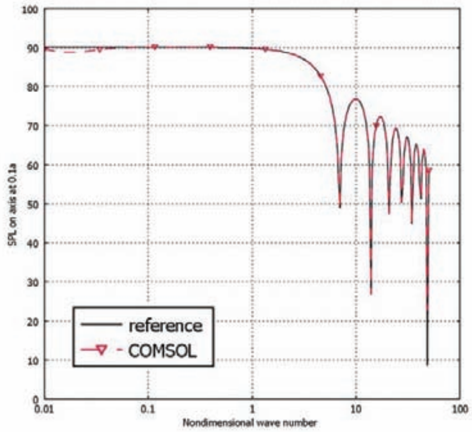

Figure 2. Sound pressure level on the axis at a distance of 0.1*a*. Note the slight deviation of the COMSOL solution from the reference at *ka*<0.1.

In the figure, results for the PML with quadratic scaling [5] and 12 layers of quadratic elements are compared to an analytical reference solution (Eq. 5-7.3, Ref. 6) in the frequency band *ka* = 0.01 – 50. The agreement between the numerical result and the reference solution is good at high frequencies, while errors are visible below *ka*=0.1. The problems at low *ka* are more visible in Fig. 3, showing the relative error in the pressure at *a*/10 for a number of different meshes.

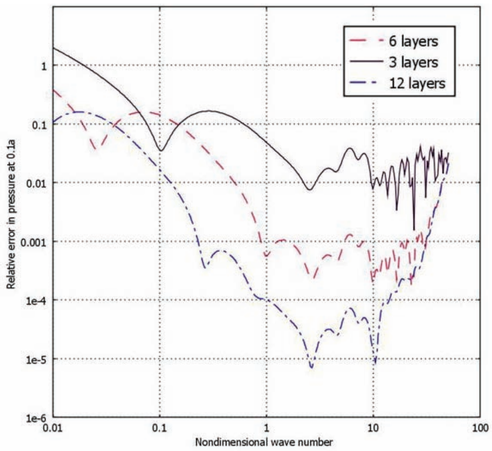

Figure 3. Logarithmic plot of the relative error as function of *ka* for PMLs with 3, 6, and 12 layers of quadratic elements.

2.2 Scattering from a rigid sphere

Similar discretization errors as those encountered in the piston radiation problem occur also in scattering problems. As an example, one can study the axisymmetric scattering of a unit-amplitude plane wave from a rigid sphere (Fig. 4).

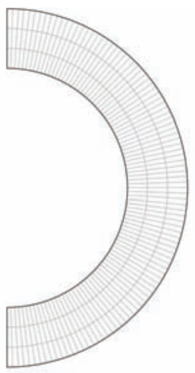

Figure 4. The rigid sphere is modeled with a PML in direct contact with the surface, here shown with a low-resolution mesh.

In Fig. 5, the backscattered pressure, sampled on the surface of the sphere, is plotted as a function of *ka*, with *a* representing the radius of the sphere. Also in this case, the agreement with an analytical reference solution, obtained from a spherical harmonic series representation [7], is very good at the higher frequencies, while discretization errors similar to those shown in Fig. 3 appear at the lower frequencies (Fig. 6).

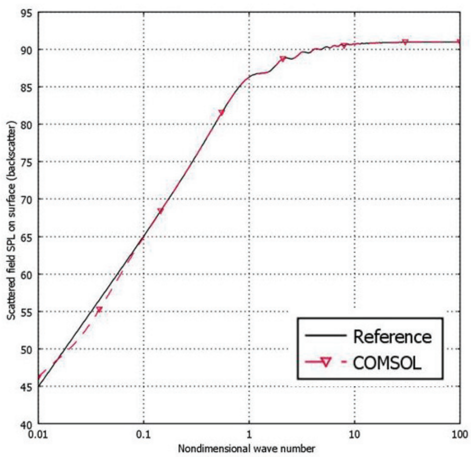

Figure 5. Backscatter sound pressure level sampled on the surface of the sphere.

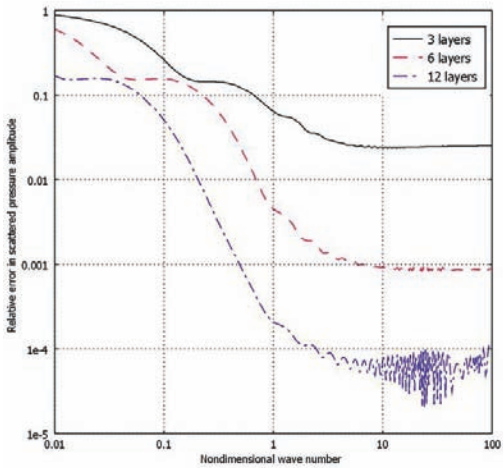

Figure 6. Relative error as a function of *ka,* for PMLs with 3, 6, and 12 layers of quadratic elements (logarithmic scale). The error is >10% at *ka*<0.1, also for the finer meshes.

3. Modified PML scaling

One way in which PML's can be defined is via a complex-valued change of coordinates [4]. In this paper, the standard approach is modified by defining a complex PML coordinate, in which the real and the imaginary part are scaled separately. The corresponding coordinate transformation, illustrated here for a PML which absorbs waves in the *x*-direction, is defined by

$$x_{\text{pml}}(s) = x_0 + x_r(s) + \frac{i}{\omega}\int_{x_0}^{s} \sigma(s')ds'$$

where $s=(x-x_0)/(x_1-x_0)$ is the dimensionless distance across the PML in the direction normal to the interface with the physical domain. This interface is located at x_0, while x_1 denotes the outer boundary of the PML. Designing an efficient PML requires a careful choice of the damping function σ and of the real part scaling function, x_r.

3.1 Unbounded scaling functions

A straightforward plane-wave analysis shows that for propagating waves, the imaginary part of the scaled coordinate is responsible for the damping of the outward travelling wave, while the real part of the scaled coordinate affects the oscillation of the solution inside the PML. In the continuous analysis, it can therefore be shown [2] that a rapidly growing imaginary part of the scaled coordinate leads to an efficient PML. The scaling of the real part does not affect the damping of the propagating wave components.

Based on this observation, Bermudez *et al.* suggest a damping function σ proportional to $1/(x_1-x)$, which corresponds to an imaginary part of the scaling function x_{pml} proportional to the logarithm of (1-s). This leads to a PML formulation which is perfectly nonreflecting for propagating waves, provided that the mesh resolution can be increased arbitrarily.

3.2 Low-*ka* corrections

A similar analysis applied to evanescent waves shows that the decay of the wave inside the PML is affected by the real part of the scaling function, while the imaginary part of the scaling gives rise to spurious anti-causal waves which must be absorbed to avoid errors in the solution. Hence, it is to be expected that the real part of the scaling function, x_r, becomes important when the features of the particular problem induce substantial evanescent components. In particular, this tends to happen at low *ka*, where the PML needs to resolve details on the geometry scale, *a*, and the wave scale, *1/k*, simultaneously.

Any finite element implementation of a PML by definition has a limited resolution, which is quantified in this paper by the number of element layers through the PML thickness. Choosing an efficient scaling function for low *ka* requires distributing the available resolution between the different scales in the pressure field. A scaling whose real and imaginary parts grow slowly as functions of the nondimensional PML coordinate, *s*, effectively moves the inner element layers closer to the physical domain, thereby increasing the resolution of the short scales.

The errors at low *ka*, illustrated in Section 2, are caused by the scaling described in [3,5] and built into COMSOL Multiphysics:

$$x_{\mathrm{pml}} = x_0 + (1-i)\frac{c}{f}s^n$$

with n=2. This scaling does not supply sufficient resolution to the rapidly decaying components of the wave field. Increasing the exponent *n* improves the result for low *ka*, but tends to decrease the accuracy for medium to high frequencies, since the resolution power at the propagating wave scale decreases in this case.

3.3 A combined approach

The above observations suggest a scaling function which combines the benefits of the polynomial, wavelength-independent scaling of [3,5], but enhanced with a *ka*-dependent exponent *n*, with the unbounded imaginary part suggested by Bermudez *et al.* [2]. Numerical experiments suggest the form

$$x_{\mathrm{pml}} = x_0 + A\frac{c}{f}\left(s^{n_{\mathrm{r}}} + i\,\log_2\left(1-s^{n_{\mathrm{i}}}\right)\right)$$

where

$$n_{\mathrm{r}} = \begin{cases} 1-\log_{10} ka, & ka<1 \\ 1, & ka\geq 1 \end{cases}$$

$$n_{\mathrm{i}} = \begin{cases} 1-\log_{10}\dfrac{ka}{10}, & ka<10 \\ 1, & ka\geq 10 \end{cases}$$

and A=0.25 is a weakly problem-dependent parameter.

4. Numerical experiments

The suggested scaling function was implemented as a “User defined“ PML scaling in the COMSOL Multiphysics Acoustics Module. A structured mesh like the ones shown in Fig. 1 and 3 was used in all cases, together with second order Lagrange shape functions.

For the baffled piston, a combination of a cylindrical PML in the radial direction and a Cartesian PML in the axial direction was chosen. As Fig. 7 shows, the suggested scaling function can produce a consistently accurate solution for low to medium frequencies. Comparing to the scaling suggested by [3,5], the proposed method gives almost an order of magnitude smaller error at the same computational cost for ka<10.

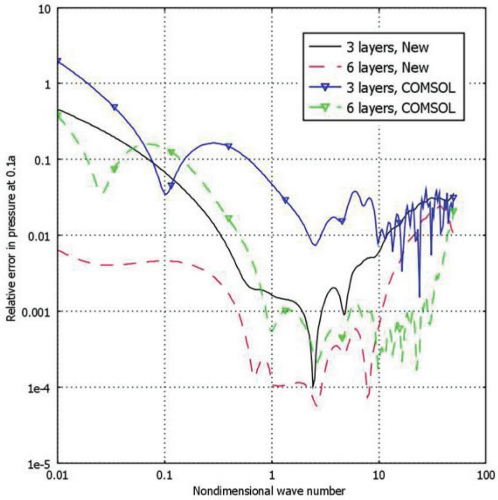

Figure 7. For the piston problem, the suggested scaling function reduces the error at low to mid frequencies considerably. Note, however, the rapid growth of the relative error above *ka*=10 for the new scaling and 6 element layers.

However, for *ka*>10, the relative error grows rapidly with increasing frequency and in the region 10<*ka*<20 it exceeds the error produced by the standard scaling by an order of magnitude. A more detailed analysis of the frequency band 10<*ka*<200 shows that the error using COMSOL's standard PMLs starts to increase rapidly around *ka*=30 and from there follows the same trend as the error produced using the modified scaling suggested here. The authors believe that these errors are caused by evanescent wave field components with wavelengths larger than *c*/*f*. The damping of such waves is limited by the maximum value of the real part of the scaling function, which is equal to *A* in the proposed scaling, but equal to 1 for the polynomial scaling. Initial experiments support this reasoning, showing that increasing *A* moves the point where the error starts to rise towards higher *ka*.

The results from the scattering case, shown in Fig. 8, are unambiguous. The suggested scaling performs better than the standard fixed-exponent polynomial scaling over the entire frequency range. In particular, the error obtained with only 3 element layers is well below 1% for *ka*>1.

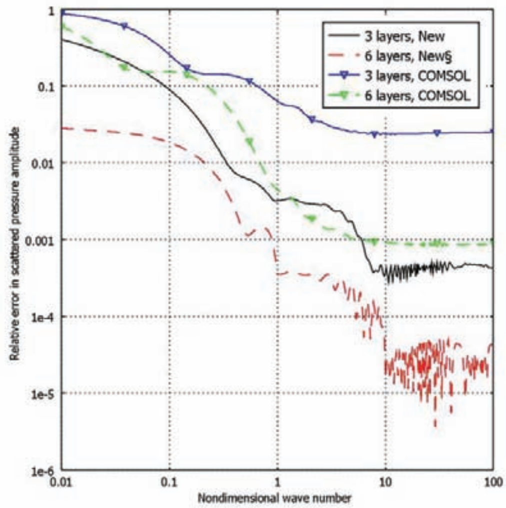

Figure 8. For the scattering problem, the suggested scaling and 3 element layers produces results comparable to the *n*=2 polynomial scaling with 6 layers.

5. Conclusions

The suggested combined scaling approach promises higher accuracy at lower computational cost for general simulations with low to moderate *ka*, as well as for higher-frequency scattering problems. Further research is needed to detect a priori which problems require a larger real part of the scaling function. The possibility of a frequency-adaptive or unbounded scaling of the real part is also being considered as a possibility for future work.

6. References

1. J.P. Bérenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys.*, **114**, 185–200 (1994).

2. A. Bermudez, L. Hervella-Nieto, A. Prieto, R. Rodriguez, "An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems," *J. Comput.Phys.*, **223**, 469 – 488, (2007).

3. M. Zampolli, A. Tesei, F.B. Jensen, N. Malm, J.B. Blottman, "A computationally efficient finite element model with perfectly matched layers applied to scattering from axially symmetric objects," *J. Acoust. Soc. Am.*, **122**, 1472–1485 (2007).

4. F. Collino and P. Monk, “The perfectly matched layer in curvilinear coordinates,” *SIAM J. Sci. Comput.*, **19**, 2061 – 2090 (1998).
5.COMSOL AB, *Comsol Multiphysics Acoustics Module, User’s Manual*, (2007).
6. A.D. Pierce, *Acoustics – An Introduction to Its Physical Principles and Applications*, Acoustical Society of America, Melville NY (1991).
7. J.J. Bowman, T.B.A. Senior, P.L.E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, North-Holland Publ. Co. (1969).