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Title:

Fatigue Damage Evaluation on Mechanical Components Under Multiaxial Loadings

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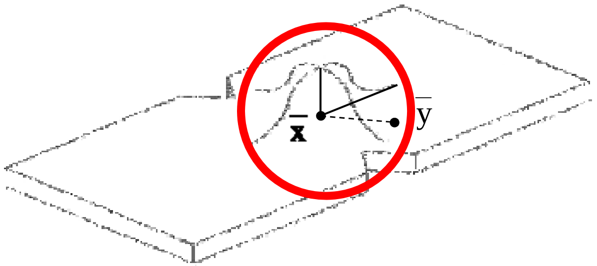
CONTENTS

AIM

1. Fatigue strength assessment of mechanical components.
2. Taking into account the effect on the fatigue strength due to the presence of complex three-dimensional (3D) notches (*gradient effect*).
3. Considering the multiaxial effect caused by external loadings as well as by multiaxial stress fields due to severe stress raisers (*multiaxial effect*).
4. Developing a numerical tool in conjunction with three-dimensional modelling tools to be used by industrial engineers.
5. Comparing theoretical fatigue estimations to experimental data in order to validate the developed approach.

Non-local model $\overset{\cong}{\sigma}(\bar{x}) = \frac{1}{V_r(\bar{x})}\int_V \alpha(\bar{x},\bar{y})\cdot\bar{\bar{\sigma}}(\bar{y})\,d\bar{y}$

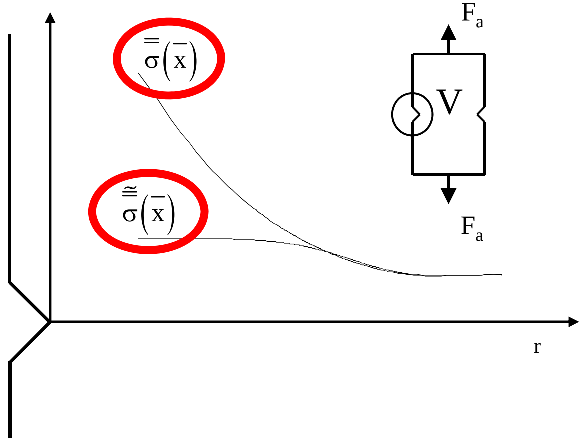
$\bar{\bar{\sigma}}(\bar{y})$ ⟶ Local stress tensor in $\bar{y}$



$\alpha(\bar{x},\bar{y}) = e^{-\frac{k^2\|\bar{x}-\bar{y}\|^2}{l^2}}$ ⟶ Weight function

$V_r(\bar{x}) = \int_V \alpha(\bar{x},\bar{y})\,d\bar{y}$ ⟶ Reference volume

$\overset{\cong}{\sigma}(\bar{x})$ ⟶ Non-local stress tensor in $\bar{x}$



From integral equation to partial differential equation

$$\tilde{\bar{\bar{\sigma}}}(\bar{x}) = \frac{1}{V_r(\bar{x})}\int_V \alpha(\bar{x},\bar{y})\cdot\bar{\bar{\sigma}}(\bar{y})\,d\bar{y} \quad (1)$$

- It is important to observe that any scalar stress field defined in volume V as a function of stress tensor $\bar{\bar{\sigma}}$ can be used as equivalent scalar ζ in equation (1).

$$\tilde{\zeta}(\bar{x}) = \frac{1}{V_r(\bar{x})}\int_V \alpha(\bar{x},\bar{y})\cdot\zeta(\bar{y})\,d\bar{y} \quad (2)$$

- Integral definition of non-local scalar quantities (2) can be approximated by a second order partial differential equation (3):

$$\tilde{\zeta}(\bar{x}) - c^2\nabla^2\tilde{\zeta}(\bar{x}) \cong \zeta(\bar{x}) \quad (3)$$

- The equation (2) turns out to be equal an implicit differential formulation (3), i.e. so-called *implicit gradient approach.*

$\nabla\tilde{\zeta}\cdot\bar{n} = 0$ ⟶ Neumann's boundary conditions, where $\bar{n}$ denotes the normal to the surface of V

c ⟶ Diffusive length related to the relevant material properties

Non-local model in fatigue

$$\tilde{\zeta}(\bar{x}) - c^2 \nabla^2 \tilde{\zeta}(\bar{x}) \cong \zeta(\bar{x})$$

- Scalars ζ and $\tilde{\zeta}$ can be different in order to consider different problems:

$\zeta, \tilde{\zeta}$ ⇒ ε_{ij} ⇒ Elasto-plastic model, low cycle fatigue

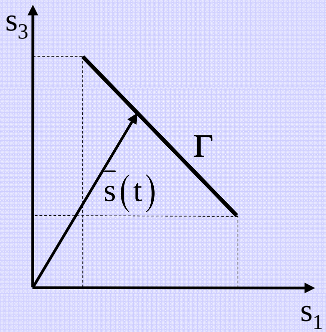
$\zeta, \tilde{\zeta}$ ⇒ $\sigma_a, \Delta\sigma_{eq}, \ldots$ ⇒ High-cycle fatigue damage evaluations

$$\tilde{\sigma}_{a,eq}(\bar{x}) - c^2 \nabla^2 \tilde{\sigma}_{a,eq}(\bar{x}) \cong \sigma_{a,eq}(\bar{x})$$

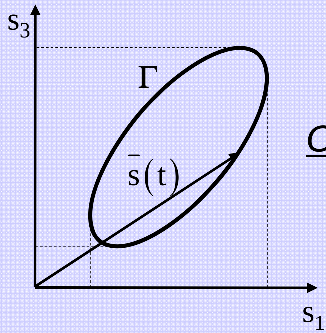
- Under non-proportional loading the principal stress directions are not constant.
- To correctly define the equivalent deviatoric component, it could be necessary to apply a multiaxial criterion together with the stress gradient approach under mixed-mode loadings.

Time variability of deviatoric component

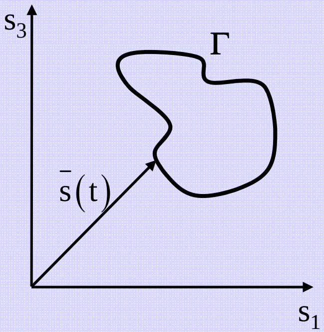
- As an external load is applied, the tip of the vector $\bar{s}(t)$ describes a curve called deviatoric stress component loading path Γ



In-phase loading



Out-of-phase loading



Variable amplitude loading

- Usually the amplitude is calculated as the maximum difference of vector $\bar{s}(t)$ at two different time instants

$$\sqrt{J_{2,a}} = \frac{1}{2}\left|\bar{s}(t_2) - \bar{s}(t_1)\right|$$

PbP approach

A. Cristofori, L. Susmel, R. Tovo, A stress invariant based criterion to estimate fatigue damage under multiaxial loading. *International Journal of Fatigue* 30 (2008), 1646-58.

Fundamental hypothesis:

The fatigue damage due to the applied loading path Γ can be estimated from the projection $\Gamma_{p,i}$ of the loading path on a convenient reference frame.

Procedure:

- Assessment of the projection reference frame
- Definition of equivalent amplitude of the deviatoric stress component

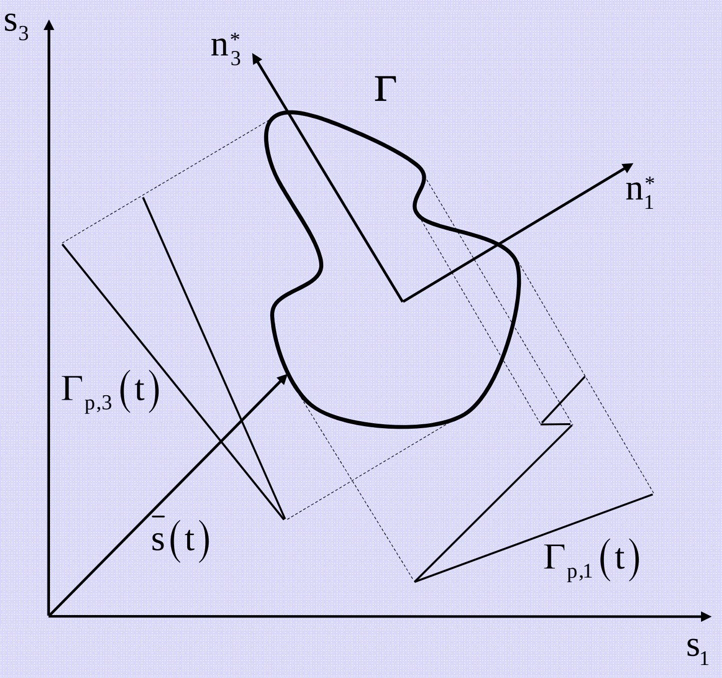
$$\sigma_{d,a} = \sqrt{\sum_i \left(\sigma_{d,a}\right)_i^2}$$

- Formulation of the multiaxial high-cycle criterion

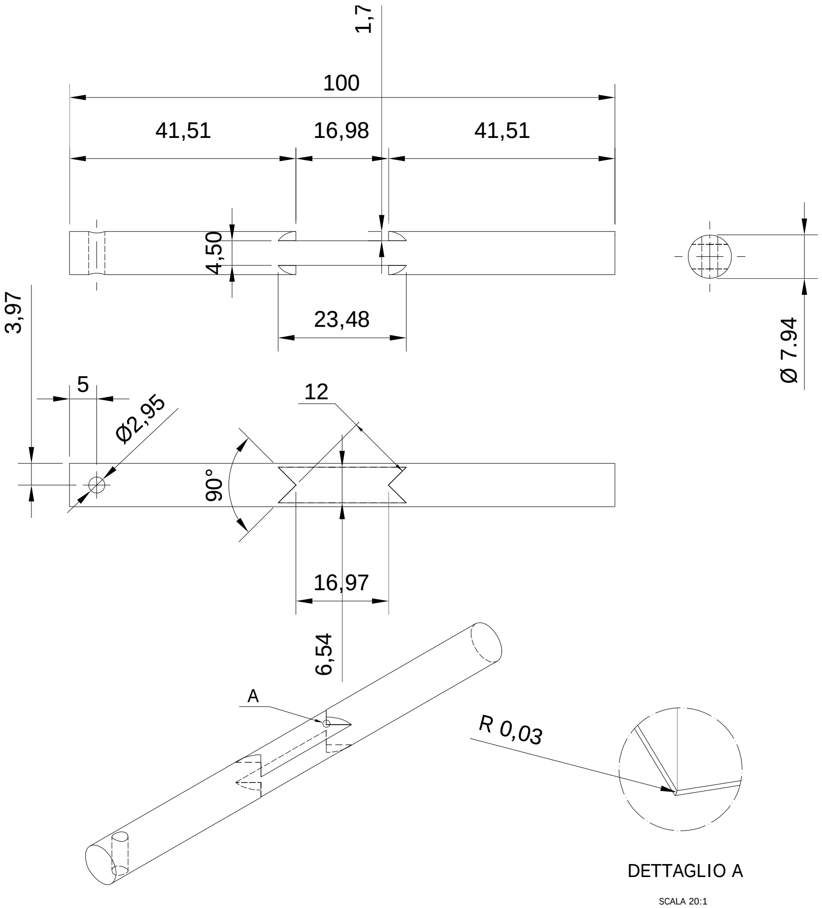
$$\sigma_{d,A}\big|_{\rho_{FL}} = \sigma_{d,A}\big|_{\rho_{FL}=0} + \rho_{FL}\cdot\left(\sigma_{d,A}\big|_{\rho_{FL}=0} - \sigma_{d,A}\big|_{\rho_{FL}=1}\right) \qquad \rho_{FL} = \sqrt{3}\cdot\frac{\sigma_{H,max}}{\sigma_{d,A}\big|_{\rho_{FL}}}$$

$$\sigma_{d,a} \le \sigma_{d,A}\big|_{\rho_{FL}}$$

Projected paths



4. EXPERIMENTAL ANALYSIS



- 70 tests were carried out by testing cylindrical 3D-notched specimens having gross diameter equal to 8mm and made of a commercial cold-rolled low-carbon steel (En3B).

- The specimens are characterized by severe stress raisers with root radius equal to 0.03mm.

- All tests were carried out in the Department of Mechanical Engineering, Trinity College of Dublin, under the supervision of prof. David Taylor.

➢Definitions:

σ_a, τ_a ⟶ Tensile and torsional stress amplitude refered to the gross section

δ ⟶ Biaxial load ratio defined as (σ_a / τ_a)

φ ⟶ Phase angle between load components

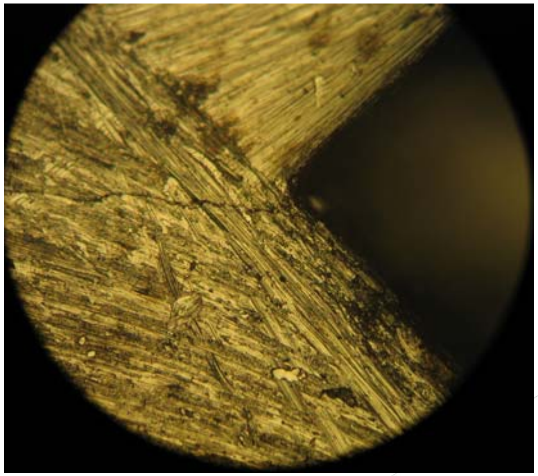
R ⟶ Nominal load ratio

➢Experimental details:

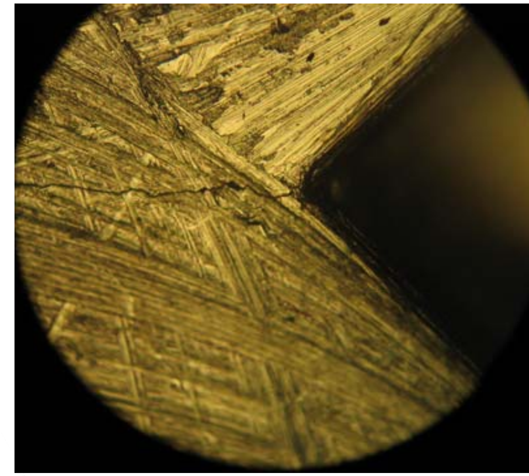
- One series of tests under pure tension fatigue loading, R=-1.
- One series of tests under pure torsion fatigue loading, R=-1.
- Two series of tests under combined tension–torsion fatigue loading, constant biaxial load ratio $\delta=\sqrt{3}$, R=-1, phase angle φ=0° or 90°
- Two series of tests under combined tension–torsion fatigue loading, constant biaxial load ratio $\delta=1$, R=-1, phase angle φ=0° or 90°

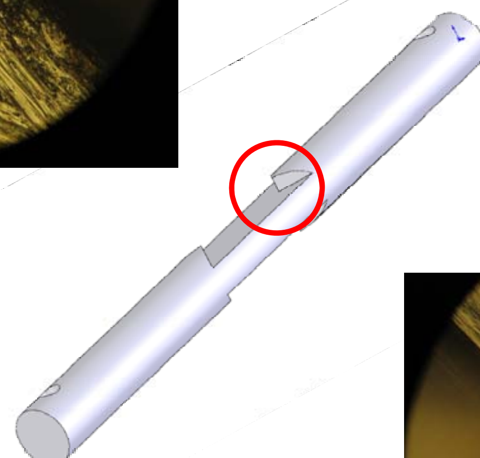
Crack paths

Torsional loading
Multiaxial loading

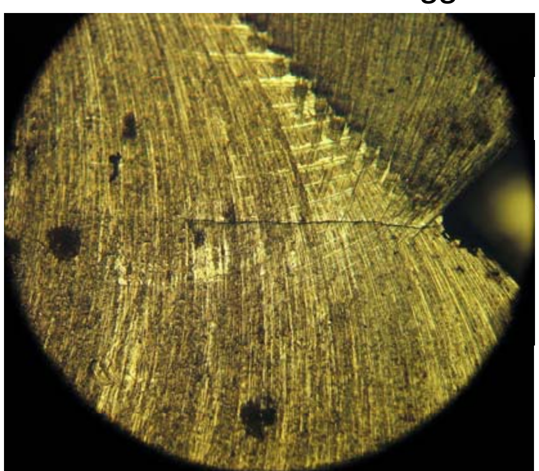


Torsional loading
Multiaxial loading

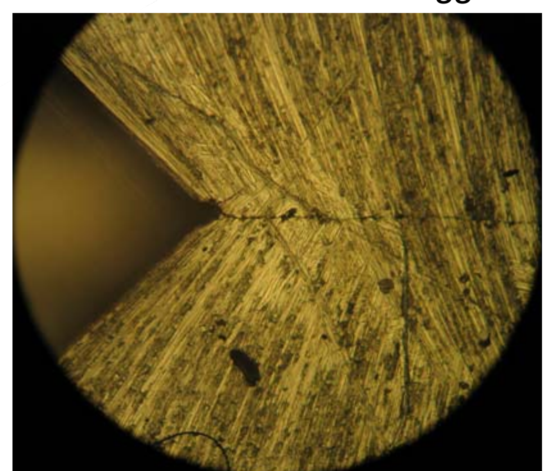




Torsional loading
Multiaxial loading

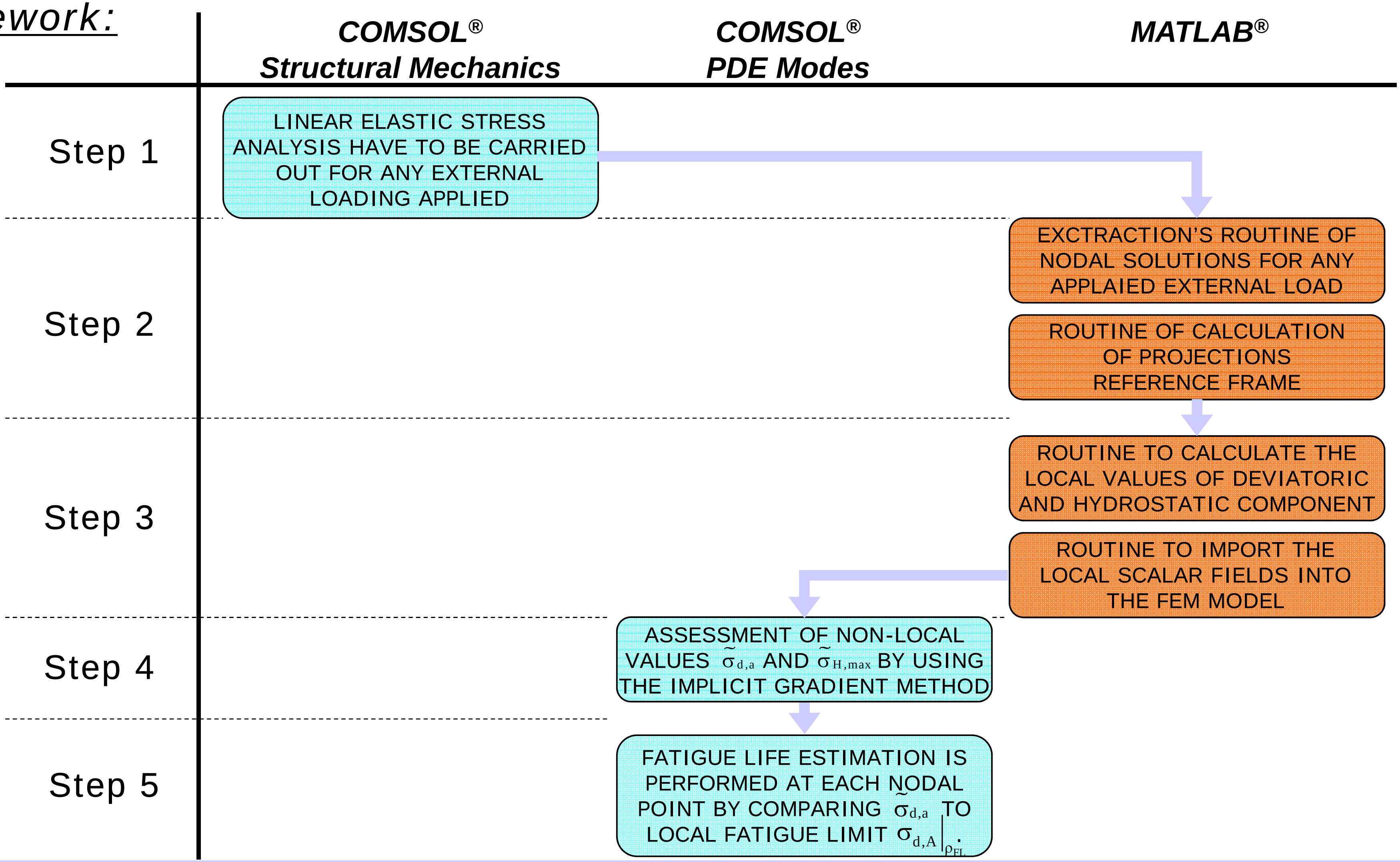


Torsional loading
Multiaxial loading



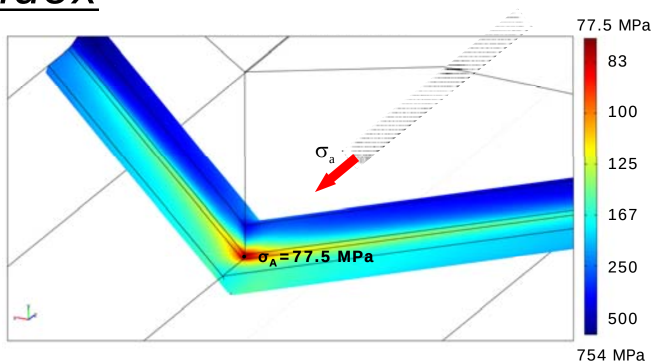
5. PROCEDURE AND TOOLS OF CALCULATIONS

The procedure framework:

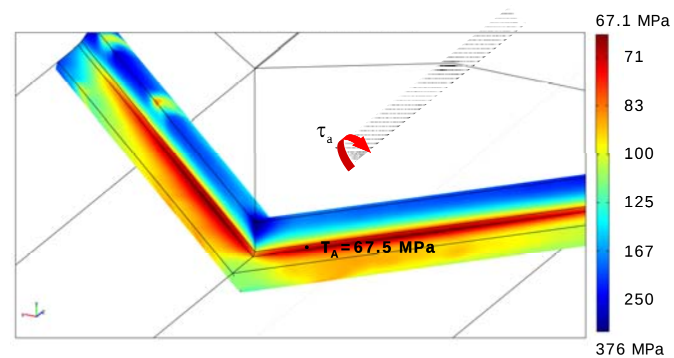


Damage index

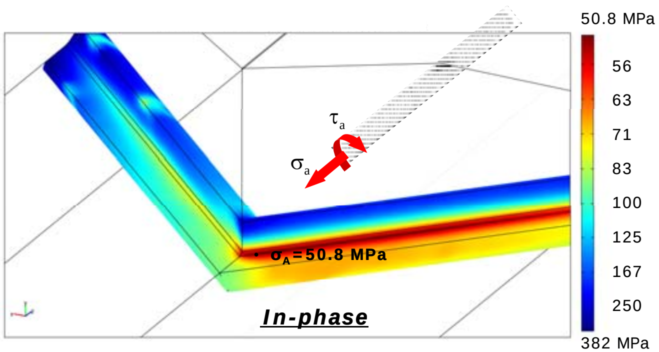
Uniaxial loading



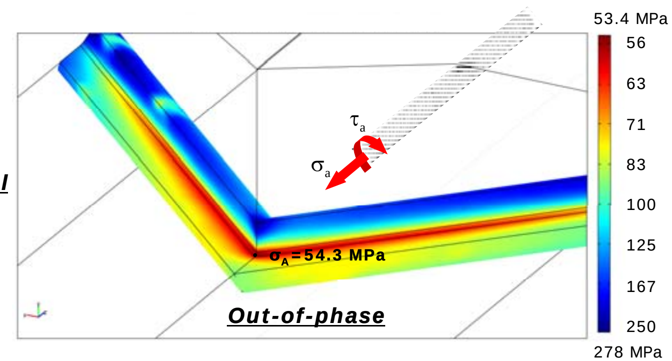
Torsional loading



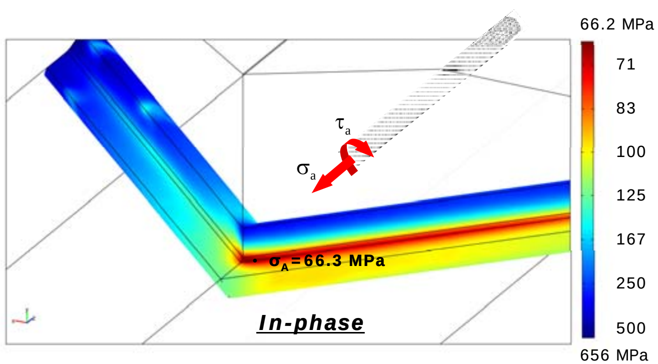
Multiaxial loading
$(\sigma_a/\tau_a)=1$



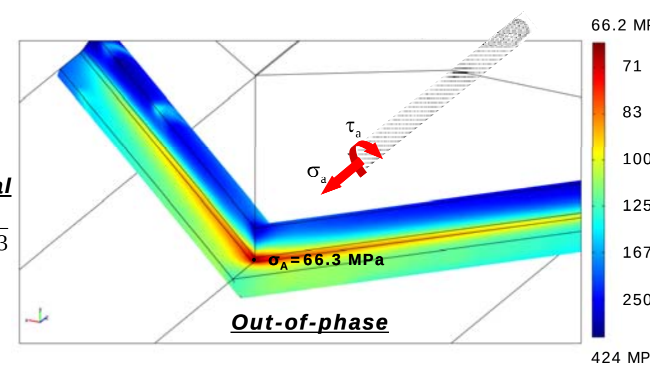
Multiaxial loading
$(\sigma_a/\tau_a)=1$



Multiaxial loading
$(\sigma_a/\tau_a)=\sqrt{3}$



Multiaxial loading
$(\sigma_a/\tau_a)=\sqrt{3}$



Damage index

GEOMETRY	LOAD CONDITIONS				EXPERIMENTAL RESULTS		FATIGUE STRENGTH PREDICTION		ERROR INDEX	CRITICAL POINT LOCATION		
	σ_a [Mpa]	τ_a [Mpa]	φ	R	σ_A [Mpa]	τ_A [Mpa]	σ_A [Mpa]	τ_A [Mpa]	E(%)	x [mm]	y [mm]	z [mm]
	1	0	0°	-1	75.7	–	77.5	–	**-2.3%**	0.00	0.00	0.00
	0	1	0°	-1	–	66.1	–	67.5	**-2.1%**	0.84	0.00	0.85
	1	1	0°	-1	52.7	–	50.8	–	**3.7%**	-0.25	0.00	0.25
	1	1	90°	-1	52.9	–	54.3	–	**-2.6%**	-0.00	0.02	0.04
	1.73	1	0°	-1	68.7	–	66.3	–	**3.6%**	-0.03	0.01	0.06
	1.73	1	90°	-1	66.7	–	66.3	–	**0.6%**	-0.00	0.02	0.04

7. CONCLUSIONS

- The devised approach is seen to be highly accurate in estimating high-cycle fatigue damage in mechanical components without the need for assuming a priori the position of the critical point.
- This approach is capable of efficiently taking into account the presence of both multiaxial loading and non zero out-of-phase angles.
- The implicit gradient method applied in conjunction with *PbP* approach proved to be a powerful engineering tool capable of efficiently designing complex, i.e. three-dimensional (3D), stress concentrations against multiaxial fatigue.
- The fatigue life estimation technique proposed in the present work is suitable for being used in situations of practical interest by directly post-processing simple linear-elastic FE models.
- Even if the results obtained so far are very satisfactory, more work needs to be done for a complete validation of this method.

Thank you for your attention

General form for multiaxial fatigue limit criteria

$$f\left(T, N\right) \leq \lambda$$

Material parameter that characterizes the fatigue limit condition (λ)

Shear stress component (T)

Normal stress component (N)

	Shear stress component	Normal stress component
Critical plane approach	Amplitude of the shear stress on the critical plane	Mean or maximum normal stress perpendicular to critical plane
Stress-invariant based approach	Amplitude of the square root of the second invariant of the stress deviator	Mean or maximum hydrostatic stress

Definition of the stress quantities related to stress invariant approach

- Deviatoric and hydrostatic components of Cauchy's stress tensor

$$\bar{\bar{\sigma}}(t)=\begin{bmatrix}\sigma_x(t) & \tau_{xy}(t) & \tau_{xz}(t)\\ \tau_{yx}(t) & \sigma_y(t) & \tau_{yz}(t)\\ \tau_{zx}(t) & \tau_{zy}(t) & \sigma_z(t)\end{bmatrix}=\sigma_H(t)\cdot\bar{\bar{I}}+\bar{\bar{\sigma}}_d(t)$$

$$\sigma_H(t)=\frac{1}{3}\mathrm{tr}\left(\bar{\bar{\sigma}}(t)\right)$$

$$\bar{\bar{\sigma}}_d(t)=\begin{bmatrix}\frac{2\sigma_x(t)-\sigma_y(t)-\sigma_z(t)}{3} & \tau_{xy}(t) & \tau_{xz}(t)\\ \tau_{yx}(t) & \frac{2\sigma_y(t)-\sigma_x(t)-\sigma_z(t)}{3} & \tau_{yz}(t)\\ \tau_{zx}(t) & \tau_{zy}(t) & \frac{2\sigma_z(t)-\sigma_x(t)-\sigma_y(t)}{3}\end{bmatrix}$$

- Deviatoric component can be represented as a 5-element vector

$$\bar{s}(t)=\begin{bmatrix}s_1(t)\\ s_2(t)\\ s_3(t)\\ s_4(t)\\ s_5(t)\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{6}\left(2\sigma_x(t)-\sigma_y(t)-\sigma_z(t)\right)\\ \frac{1}{2}\left(\sigma_y(t)-\sigma_z(t)\right)\\ \tau_{xy}(t)\\ \tau_{xz}(t)\\ \tau_{yz}(t)\end{bmatrix}$$

- Square root of the second invariant of the stress deviator

$$\sqrt{J_2(t)}=\sqrt{\bar{s}(t)\cdot\bar{s}(t)}=\sqrt{\sum_{i=1}^{5}\left(s_i(t)\right)^2}$$

How to find the amplitude?

The projection reference frame

- Centroid of the path

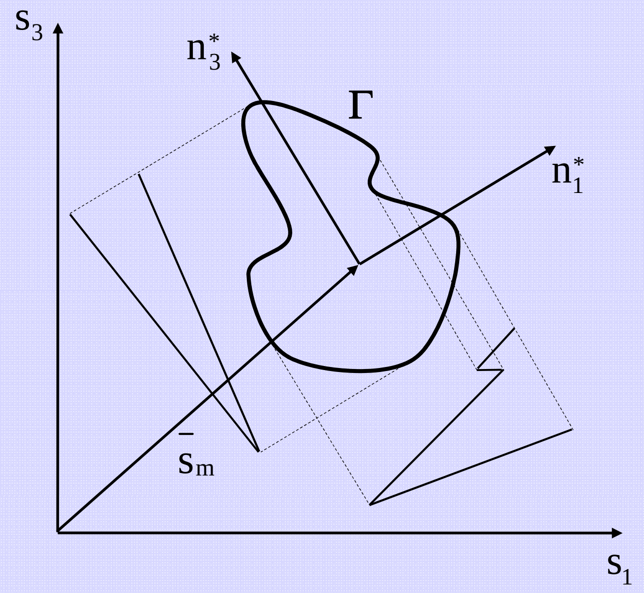
$$s_{m,i} = \frac{1}{T}\int_T s_i(t)\,dt \qquad i=1,\ldots,5$$

- Rectangular moment of inertia of the path

$$C_{ij} = \int_T \left(s_i(t) - s_{m,i}\right)\cdot\left(s_j(t) - s_{m,j}\right)dt \qquad i,j=1,\ldots,5$$

- Eigenvectors are invariant for coordinate transformation and their calculation is numerically efficient.
- The eigenvector reference frame coincides with the maximum variance reference frame.

Projection reference frame



- Fatigue damage is strongly related to the variance of the signals. For instance, under amplitude constant load the maximum damage reference frame coincides with the maximum variance reference frame.

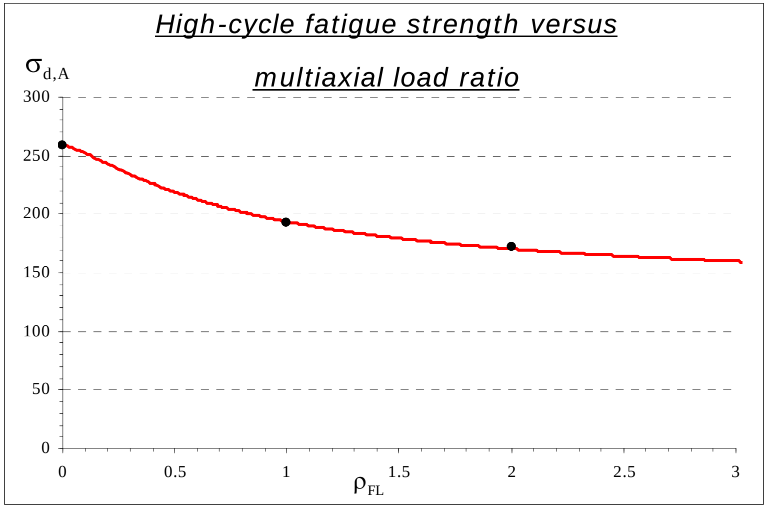
Calibration of the fatigue strength criterion

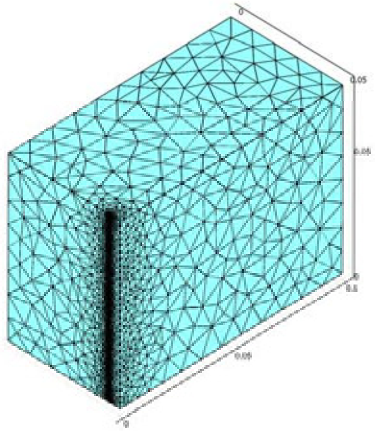
EN3B

Serie	R	ρ_{FL}	$\sigma_{d,A}\|_{\rho_{FL}}$ a $1\cdot10^6$ cicli [MPa]	$\sigma_{d,A}\|_{\rho_{FL}}$ a $2\cdot10^6$ cicli [MPa]
Monoassiale	-1	1	199.7	192.8
Monoassiale	0	2	175.1	171.9
Torsione pura	-1	0	268.6	258.6

- Relation between fatigue strength at 2 million cycle and ρ_{FL}

$$\sigma_{d,A}(\rho_{FL})[\text{MPa}] = -\left(130\cdot e^{\left(-\frac{0.77}{\rho_{FL}+0.2}\right)}\right)+262$$



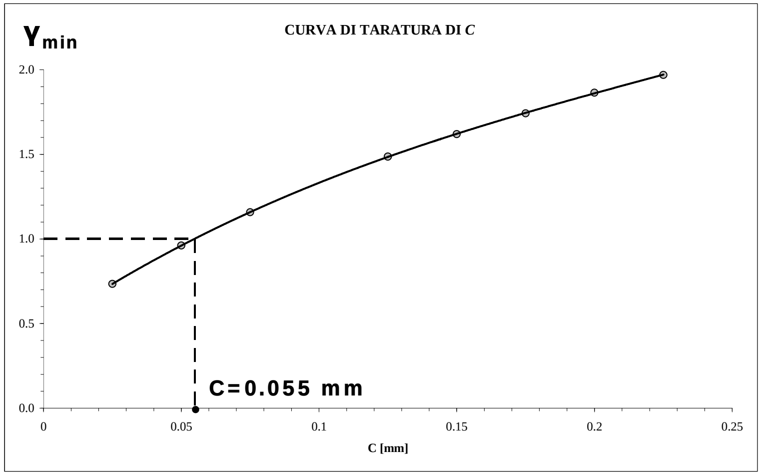


- Fatigue limit condition for a crack

$$\Delta K_I = y\cdot\Delta\sigma_{nom}\cdot\sqrt{\pi\cdot a} = \Delta K_{th}$$

- High-cycle fatigue safety factor

$$\gamma = \frac{\sigma_{d,A}\big|_{\rho_{FL}}}{\tilde{\sigma}_{d,a}}$$



The procedure framework:

1. A usual linear elastic stress analysis have to carry out for each type of external load applied to the specimen.

2. Local projection reference frame is evaluated at every mesh nodal point.

3. Local values of the amplitude of deviatoric component $\sigma_{d,a}$ and maximum hydrostatic component $\sigma_{H,max}$ are calculated at every node.

4. Non-local values $\tilde{\sigma}_{d,a}$ and $\tilde{\sigma}_{H,max}$ are calculated at each nodal point by using the implicit gradient method:

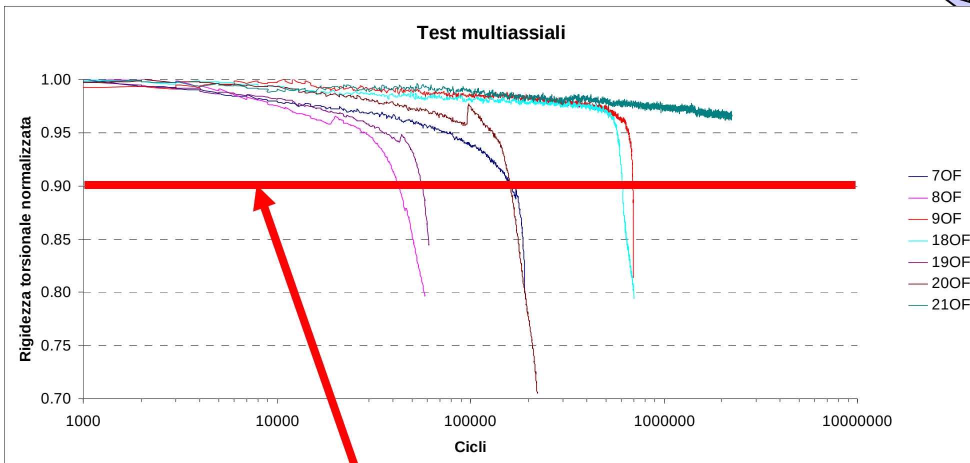
$$\tilde{\sigma}_{d,a} - c^2\nabla^2\tilde{\sigma}_{d,a} \cong \sigma_{d,a} \qquad \nabla\left(\tilde{\sigma}_{d,a}\right)\cdot\mathbf{n} = 0$$

$$\tilde{\sigma}_{H,max} - c^2\nabla^2\tilde{\sigma}_{H,max} \cong \sigma_{H,max} \qquad \nabla\left(\tilde{\sigma}_{H,max}\right)\cdot\mathbf{n} = 0$$

5. Multiaxial load ratio ρ_{FL} can be calculated as: $\rho_{FL} = \sqrt{3}\cdot\dfrac{\tilde{\sigma}_{H,max}}{\tilde{\sigma}_{d,a}}$

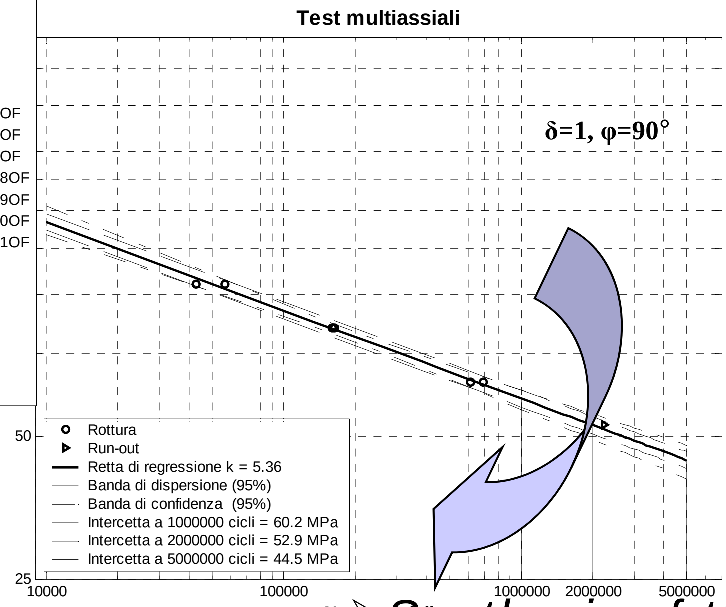
6. Fatigue life estimation is finally performed at each nodal point comparing $\tilde{\sigma}_{d,a}$ with local fatigue limit $\sigma_{d,A}\big|_{\rho_{FL}}$ calculated as function of ρ_{FL}.

➢*Fatigue tests*



Failure criterion

➢*Statistical analysis*

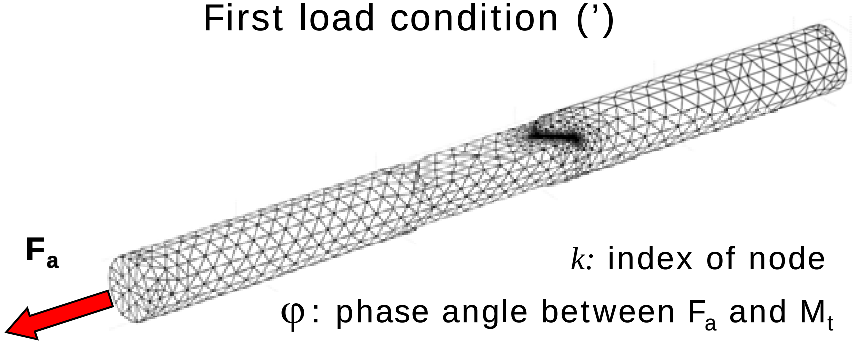


Serie	Nr. provini	R	δ	φ	k	σ a 1·1 [M		
Monoassiale	15	-1	-	-	3.76	91.0	**75.7**	59.4
Torsione pura	16	-1	-	-	6.87	73.2	**66.1**	57.9
Multiassiale	13	-1	1.73	0°	4.98	79.0	**68.7**	57.2
Multiassiale	11	-1	1.73	90°	4.61	77.5	**66.7**	54.7
Multiassiale	8	-1	1	0°	5.67	59.6	**52.7**	44.8
Multiassiale	7	-1	1	90°	5.36	60.2	**52.9**	44.5

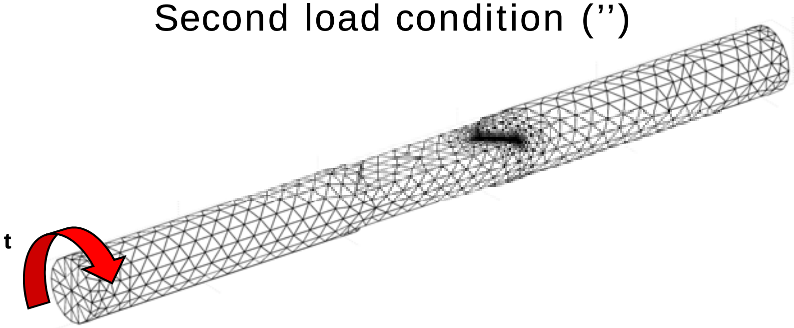
➢*Synthesis of the experimental results generated under uniaxial and multiaxial loads*

Definition of the stress quantities

First load condition (')



Second load condition ('')



k: index of node

φ: phase angle between F_a and M_t

$$\sigma'(t,k)=\begin{bmatrix}\sigma'_x(t,k) & \tau'_{xy}(t,k) & \tau'_{xz}(t,k)\\ \tau'_{xy}(t,k) & \sigma'_y(t,k) & \tau'_{yz}(t,k)\\ \tau'_{xz}(t,k) & \tau'_{yz}(t,k) & \sigma'_z(t,k)\end{bmatrix}$$

$$\sigma''(t,k)=\begin{bmatrix}\sigma''_x(t,k) & \tau''_{xy}(t,k) & \tau''_{xz}(t,k)\\ \tau''_{xy}(t,k) & \sigma''_y(t,k) & \tau''_{yz}(t,k)\\ \tau''_{xz}(t,k) & \tau''_{yz}(t,k) & \sigma''_z(t,k)\end{bmatrix}$$

$$s_1(t)=\frac{1}{2\sqrt{3}}\left(2\sigma_x(t)-\sigma_y(t)-\sigma_z(t)\right)=$$

$$=\frac{1}{2\sqrt{3}}\left(\begin{matrix}2\left(\sigma'_{x,m}+\sigma'_{x,a}\sin(\omega t)+\sigma''_{x,m}+\sigma''_{x,a}\sin(\omega t-\varphi)\right)-\sigma'_{y,m}-\sigma'_{y,a}\sin(\omega t)-\\ -\sigma''_{y,m}-\sigma''_{y,a}\sin(\omega t-\varphi)-\sigma'_{z,m}-\sigma'_{z,a}\sin(\omega t)-\sigma''_{z,m}-\sigma''_{z,a}\sin(\omega t-\varphi)\end{matrix}\right)$$

$$s_2(t)=\frac{1}{2}\left(\sigma_y(t)-\sigma_z(t)\right)=$$

$$=\frac{1}{2}\left(\begin{matrix}\sigma'_{y,m}+\sigma'_{y,a}\sin(\omega t)+\sigma''_{y,m}+\sigma''_{y,a}\sin(\omega t-\varphi)-\\ -\sigma'_{z,m}-\sigma'_{z,a}\sin(\omega t)-\sigma''_{z,m}-\sigma''_{z,a}\sin(\omega t-\varphi)\end{matrix}\right)$$

$$s_3(t)=\tau_{xy}(t)=\left(\tau'_{xy,m}+\tau'_{xy,a}\sin(\omega t)+\tau''_{xy,m}+\tau''_{xy,a}\sin(\omega t-\varphi)\right)$$

$$s_4(t)=\tau_{xz}(t)=\left(\tau'_{xz,m}+\tau'_{xz,a}\sin(\omega t)+\tau''_{xz,m}+\tau''_{xz,a}\sin(\omega t-\varphi)\right)$$

$$s_5(t)=\tau_{yz}(t)=\left(\tau'_{yz,m}+\tau'_{yz,a}\sin(\omega t)+\tau''_{yz,m}+\tau''_{yz,a}\sin(\omega t-\varphi)\right)$$

$$\overline{s'}(t)=\left[s'_1(t)\ \vert\ s'_2(t)\ \vert\ s'_3(t)\ \vert\ s'_4(t)\ \vert\ s'_5(t)\right]$$

$$\overline{s''}(t)=\left[s''_1(t)\ \vert\ s''_2(t)\ \vert\ s''_3(t)\ \vert\ s''_4(t)\ \vert\ s''_5(t)\right]$$

$$\overline{s}(t)=\overline{s'}(t)+\overline{s''}(t)=\left[s_1(t)\ \vert\ s_2(t)\ \vert\ s_3(t)\ \vert\ s_4(t)\ \vert\ s_5(t)\right]$$

5. PROCEDURE AND TOOLS OF CALCULATIONS

The maximum variance reference frame

- Variance and covariance terms of $\bar{s}(t)$

$$C_{ij} = \int_T \left(s_i(t) - s_{i,m}\right) \cdot \left(s_j(t) - s_{j,m}\right) dt \qquad i,j = 1,\dots\dots,5$$

- Covariance matrix is a symmetric square matrix of order 5

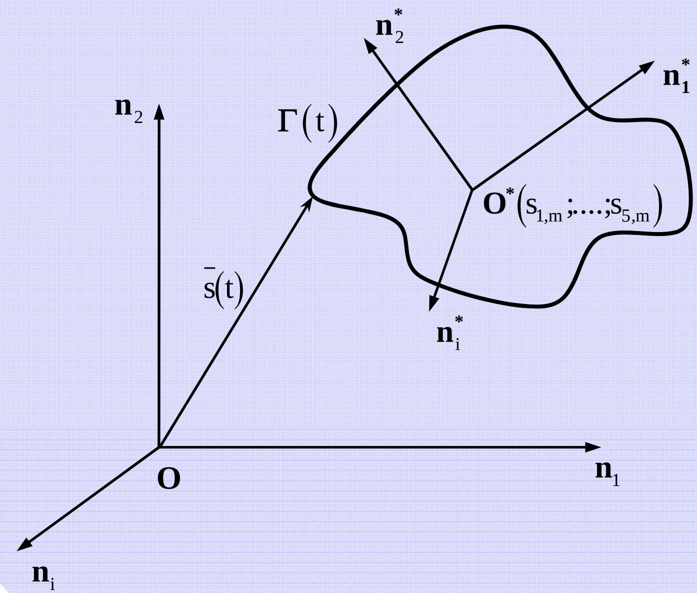
$$\overline{\overline{C}}_k = \begin{bmatrix} C_{11,k} & \cdots & C_{1j,k} \\ \vdots & \ddots & \vdots \\ C_{i1,k} & \cdots & C_{ij,k} \end{bmatrix} \qquad \begin{matrix} i,j = 1,\dots\dots,5 \\ k = 1,\dots\dots,n \end{matrix}$$

- The covariance matrix have to be diagonalised in order to find the direction of maximum variance

$$\overline{\overline{C}}_k = \overline{\overline{N}} \cdot \overline{\overline{\lambda}} \cdot \overline{\overline{N}}^{-1}$$

$$\overline{\overline{N}} = \begin{bmatrix} \overline{n}_1^* & \cdots & \overline{n}_5^* \end{bmatrix}_k$$

Projection reference frame



$$C_{11,k} = \frac{1}{T}\int_0^T \left(s_1(t) - s_{1,m}\right)^2 dt =$$

$$= \frac{1}{24}\begin{pmatrix} 4\left(\sigma'_{x,a}\right)^2 + 4\left(\sigma''_{x,a}\right)^2 + \left(\sigma'_{y,a}\right)^2 + \left(\sigma''_{y,a}\right)^2 + \left(\sigma'_{z,a}\right)^2 + \left(\sigma''_{z,a}\right)^2 + 8\sigma'_{x,a}\sigma''_{x,a}\cos\varphi - \\ -4\sigma'_{x,a}\sigma'_{y,a} - 4\sigma''_{x,a}\sigma''_{y,a} - 4\sigma'_{x,a}\sigma'_{y,a}\cos\varphi - 4\sigma''_{x,a}\sigma'_{y,a}\cos\varphi - 4\sigma'_{x,a}\sigma'_{z,a} - \\ -4\sigma'_{x,a}\sigma''_{z,a}\cos\varphi - 4\sigma''_{x,a}\sigma'_{z,a}\cos\varphi - 4\sigma''_{x,a}\sigma''_{z,a} + 2\sigma'_{y,a}\sigma''_{y,a}\cos\varphi + \\ +2\sigma'_{y,a}\sigma'_{z,a} + 2\sigma'_{y,a}\sigma''_{z,a}\cos\varphi + 2\sigma''_{y,a}\sigma'_{z,a}\cos\varphi + 2\sigma''_{y,a}\sigma''_{z,a} + 2\sigma'_{z,a}\sigma''_{z,a}\cos\varphi \end{pmatrix}$$

$$C_{12,k} = C_{21,k} = \frac{1}{T}\int_0^T \left(s_1(t) - s_{1,m}\right)\left(s_2(t) - s_{2,m}\right) dt =$$

$$= \frac{1}{8\sqrt{3}}\begin{pmatrix} \left(\sigma'_{z,a}\right)^2 + \left(\sigma''_{z,a}\right)^2 - \left(\sigma'_{y,a}\right)^2 - \left(\sigma''_{y,a}\right)^2 + 2\sigma'_{x,a}\sigma'_{y,a} + 2\sigma'_{x,a}\sigma''_{y,a}\cos\varphi - \\ -2\sigma'_{x,a}\sigma'_{z,a} - 2\sigma'_{x,a}\sigma''_{z,a}\cos\varphi + 2\sigma''_{x,a}\sigma'_{y,a}\cos\varphi + 2\sigma''_{x,a}\sigma''_{y,a} - \\ -2\sigma''_{x,a}\sigma'_{z,a}\cos\varphi - 2\sigma''_{x,a}\sigma''_{z,a} - 2\sigma'_{y,a}\sigma''_{y,a}\cos\varphi + 2\sigma'_{z,a}\sigma''_{z,a}\cos\varphi \end{pmatrix}$$

Analytical formulations of deviatoric and hydrostatic stress components

$$\bar{s}(t)=\bar{s}'(t)+\bar{s}''(t)=\bar{s}'_m+\bar{s}'_a\cdot\sin(\omega t)+\bar{s}''_m+\bar{s}''_a\cdot\sin(\omega t-\varphi)$$

$$\bar{s}(t)\Big|_{\bar{n}_i^*}=\frac{\left\langle \bar{s}(t),\bar{n}_i^*\right\rangle}{\left\langle \bar{n}_i^*,\bar{n}_i^*\right\rangle}\cdot\bar{n}_i^* \qquad i=1,\dots,5$$

$$\sigma_H(t)=\sigma'_{H,m}+\sigma'_{H,a}\cdot\sin(\omega t)+\sigma''_{H,m}+\sigma''_{H,a}\cdot\sin(\omega t-\varphi)$$

- Calculation of maximum values of $\bar{s}(t)$ and $\sigma_H(t)$ functions

$$\frac{d\left(\bar{s}(t)\Big|_{\bar{n}_i^*}\right)}{dt}=0 \qquad i=1,\dots,5 \qquad\qquad \frac{d\left(\sigma_H(t)\right)}{dt}=0$$

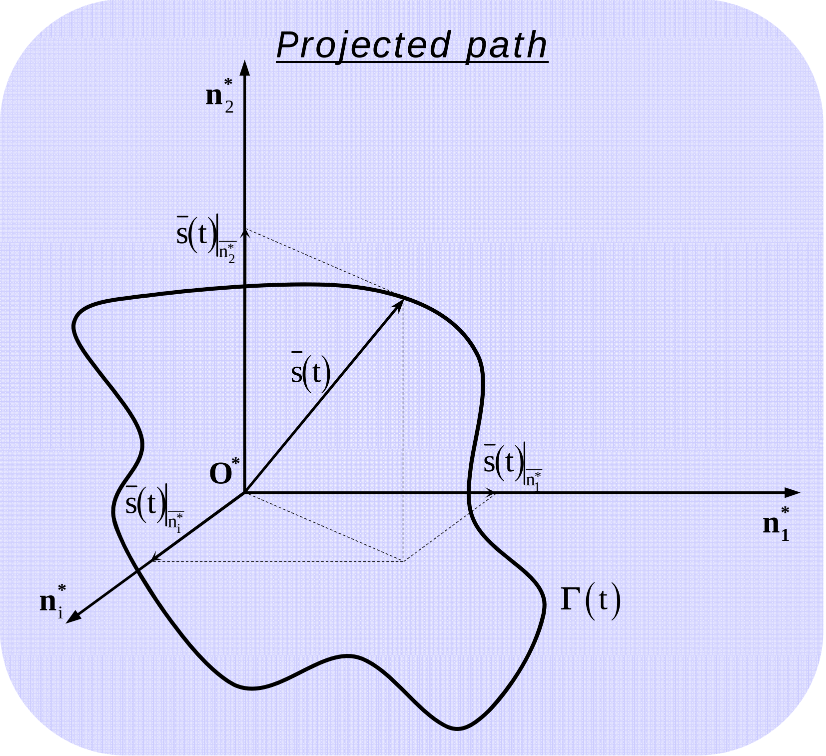
$$\alpha_{\bar{n}_i^*}=\arctan\left(-\left(\frac{\left\langle \bar{s}'_a,\bar{n}_i^*\right\rangle}{\left\langle \bar{s}''_a,\bar{n}_i^*\right\rangle}\cdot\frac{1}{\sin\varphi}+\frac{1}{\tan\varphi}\right)\right)_{i=1,\dots,5} \qquad \beta_{\bar{n}_i^*}=\alpha_{\bar{n}_i^*}+\pi$$

$$\alpha_{\sigma_H}=\arctan\left(-\left(\frac{\sigma'_{H,a}}{\sigma''_{H,a}}\cdot\frac{1}{\sin\varphi}+\frac{1}{\tan\varphi}\right)\right) \qquad \beta_{\sigma_H}=\alpha_{\sigma_H}+\pi$$

- Analytical solutions

$$\left(\bar{s}\Big|_{\bar{n}_i^*}\right)_a=\left[\left\langle \bar{s}'_a,\bar{n}_i^*\right\rangle\cdot\sin\left(\alpha_{\bar{n}_i^*}\right)+\left\langle \bar{s}''_a,\bar{n}_i^*\right\rangle\cdot\sin\left(\alpha_{\bar{n}_i^*}-\varphi\right)\right]_{i=1,\dots,5}$$

$$\sigma_{d,a}=\sqrt{\sum_i\left(\sigma_{d,a}\right)_i^2}=\sqrt{\sum_{i=1}^{5}\left(\bar{s}\Big|_{\bar{n}_i^*}\right)_a^2}$$



$$\sigma_{H,max}=\left|\sigma'_{H,a}\cdot\sin\left(\alpha_{\sigma_H}\right)+\sigma''_{H,a}\cdot\sin\left(\alpha_{\sigma_H}-\varphi\right)\right|$$