Gravity-Driven Film Flow: Design of Bottom Topography

C. Heining* and N. Aksel
Applied Mechanics and Fluid Dynamics, University of Bayreuth
*Corresponding author: University of Bayreuth, D-95440 Bayreuth, christian.heining@uni-bayreuth.de

Abstract: We study the gravity-driven film flow of a Newtonian liquid down an inclined plane. Many applications such as heat- and mass exchangers and evaporators or film coaters require undulated or rippled bottom topographies. In these cases, the interplay of gravity, surface tension and inertia leads to a response of the interface which furthermore strongly depends on the shape of the bottom topography. Instead of determining the free surface position as in the case of the direct problem, we prescribe a free surface shape and seek the corresponding bottom topography. Furthermore, we consider the stability of the system and derive conditions for the onset of instabilities at the free surface in form of surface waves.

Keywords: thin films, free surface flow, bottom topography, inverse problem.

1. Introduction

Gravity-driven film flow over topography is a fundamental problem in fluid mechanics. The basic problem of the steady flow over a flat incline has even an analytical solution found by Nusselt in 1916 [1]. A great variety of applications in many kinds of industrial and natural systems deals with inclines which are undulated. Experimental investigations show that the flow over a wavy configuration exhibits new phenomena such as surface rollers, standing waves or hydraulic jumps in form of steep shocks at the free surface [2] which cannot be found in the flow over flat inclines. Further effects which can be observed deal with modifications in the flow field like the formation of eddies [3].

A lot of theoretical, numerical and experimental results have been published in the last years dealing with the forward problem of finding the unknown free surface for a given bottom topography. The related inverse problem which consists in finding a topography that causes a target free surface is a relatively new field in research. A first approach was presented by Gramlich et al. [4] who controlled the capillary ridge in the flow over a trench by applying a heat source. Scholle et al. [5] improved the drag by modifying the topography. Sellier [6] solved the inverse problem by prescribing a free surface and solving for the topography in the lubrication limit. He refers to two main applications. The first one deals with coating defects which could be removed by controlling the free surface shape. The second one comes from an experimentalist's point of view: in the case when the flow of an opaque liquid over an unknown topography is considered, the inverse problem allows to find easily the bottom topography by measuring the free surface shape.

Sellier [6] treated the inverse problem in the lubrication approximation which is restricted to very small Reynolds numbers. In the following we present a weighted-residual integral boundary-layer approach which is valid for up to moderate Reynolds numbers. As a test case we consider the configuration of a wavy free surface and solve for the corresponding bottom topography.

The outline is as follows. We first present the modeling and the governing equations. The governing equations are then solved numerically. To test the method we also apply an analytical approximation and compare the results. Finally, we study the linear stability of the steady solution computed before by solving a corresponding eigenvalue problem.

2. Modeling

We study an incompressible viscous liquid flowing down a substrate inclined at an angle α. The free surface is given by the function

$$h(x) = d + a\sin(2\pi x/L), \qquad (1)$$

where d is the mean film thickness, a the amplitude of the free surface undulation, L its wavelength and x the spatial coordinate in downstream direction. A sketch of the problem setup is shown in Figure 1. The corresponding unknown bottom topography reads $b(x)$ and the film thickness can be computed with the relation $f = h - b$.

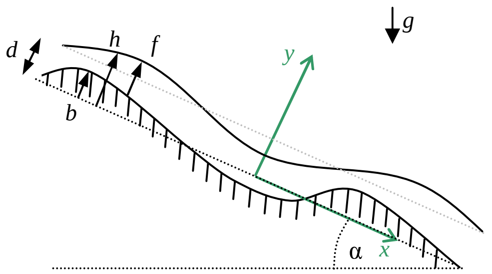


Figure 1. Sketch of the problem [7].

The flow is then completely described by the Navier-Stokes equation and the continuity equation. At the boundaries we have the no-slip and no-penetration condition at the bottom and the dynamic and kinematic boundary condition at the free surface.

In what follows we assume a thin film approximation which claims that $\delta^2 \ll 1$, where $\delta = 2\pi d / L$ is the thin-film parameter. Writing the governing equations in dimensionless form and applying the weighted-residual integral boundary-layer method we arrive at the two evolution equations for f and q

$$\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{2}$$

$$\delta \operatorname{Re}\left(\frac{\partial q}{\partial t} + \frac{17q}{7f}\frac{\partial q}{\partial x} - \frac{9q^2}{7f^2}\frac{\partial f}{\partial x}\right) - \frac{5}{2}\left(f - \frac{q}{f^2}\right) - \frac{5}{2} f\left(\delta Bo^{-1}\frac{\partial^3 h}{\partial x^3} - \delta \cot\alpha \frac{\partial h}{\partial x}\right) = 0, \tag{3}$$

where q is the local flow rate in a cross section. All variables denote dimensionless quantities. Details of the derivation can be found in [7]. The model (2), (3) will be referred to as the weighted-residual integral boundary-layer (WRIBL) in the following. The dimensionless free surface contour reads

$$h(x) = 1 + A \sin x, \tag{4}$$

where A measures the amplitude of the free surface. We summarize all dimensionless parameters in Table 1.

Table 1: Dimensionless numbers

notation	definition	physical interpretation
Re	Reynolds number	inertia
Bo^{-1}	inverse Bond number	surface tension
$\cot\alpha$	cotangens of inclination angle	hydrostatic pressure
δ	film thickness parameter	
A	waviness of the free surface	

We note that (2) and (3) are integral formulations of the mass balance and the momentum balance, respectively. We identify the following terms in (3): The first term in brackets comes from the convective terms in the Navier-Stokes equation and is responsible for the influence of inertia. The second term comes from the influence of gravity and wall shear stress and the third term is due to surface tension and hydrostatic pressure. Equations (2) and (3) are the starting point of the following investigations.

3. Numerical solution of the steady problem

We first focus on the steady problem. In the steady case (2) and (3) reduce to a single ordinary differential equation for f

$$\delta \operatorname{Re}\left(-\frac{9}{7f^2}\frac{\partial f}{\partial x}\right) - \frac{5}{2}\left(f - \frac{1}{f^2}\right) - \frac{5}{2} f\left(\delta Bo^{-1}\frac{\partial^3 h}{\partial x^3} - \delta \cot\alpha \frac{\partial h}{\partial x}\right) = 0. \tag{5}$$

We implement (5) in COMSOL by rewriting (5) into a first order differential equation and applying the PDE mode. Eq. (5) is then solved over one period of the free surface hence $x \in [0, 2\pi]$. We furthermore apply periodic boundary conditions which excludes in- and outflow effects. We note that solving for f then leads the unknown bottom topography by the relation $b = h - f$.

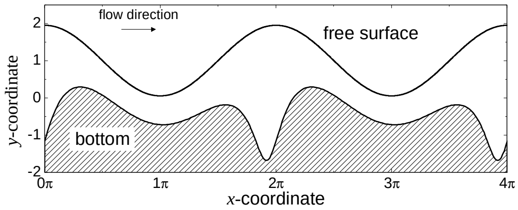


Figure 2. Free surface and bottom contour. The parameters are A=0.95, Re=90, Bo^{-1}=20, cotα=1, δ=0.2, [7].

A typical solution for the problem is shown in Figure 2 where we plot the given free surface shape and the numerical solution of the bottom topography.

Figure 2 reveals that the topography needs to contain steep troughs in order to keep the free surface at this undulated shape. For very small free surface undulations we expect that the solution for the bottom topography tends to the flat bottom in the limit case. Higher amplitudes lead to stronger deflections in the topography. Figure 3 shows a study for increasing free surface amplitude *A*.

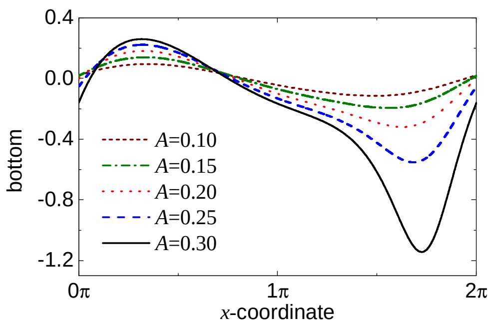


Figure 3. Shape of the bottom topography for changing free surface amplitudes. The parameters are Re=20, Bo^{-1}=20, cotα=1, δ=0.2, [7].

The previous images motivate us to consider the Fourier decomposition of the bottom as a measure for its nonlinearity. From Figure 3 we conclude that the first Fourier mode, which corresponds to the same wavelength as the free surface, is dominating when *A* is sufficiently small. Increasing the free surface amplitude leads to a generation of higher harmonics. This behavior is further investigated in Figure 4 where we display the first three harmonics of the

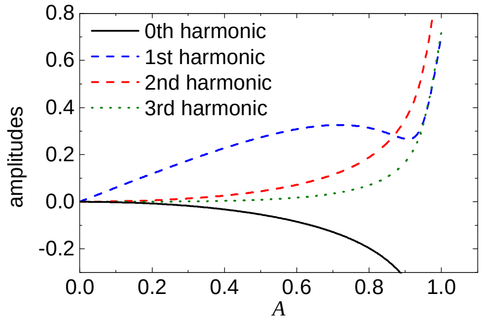


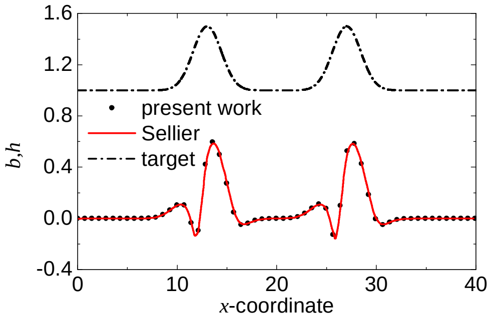
Figure 4. Fourier decomposition of the bottom topography. The parameters are A=0.95, Re=90, Bo^{-1}=20, cotα=1, δ=0.2, [7].

bottom contour. We observe that the first harmonic in fact dominates the others, however, for higher *A* the topography becomes more and more nonlinear when higher harmonics come into play. At $A \approx 0.95$, which corresponds to the case in Figure 2, we find that higher harmonics increase considerably.

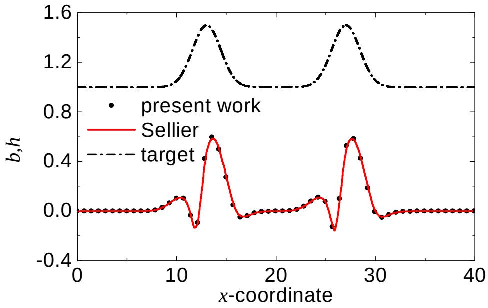
4. Validation of the numerical results

The WRIBL-model developed in the second section allows for the description of inertial flows up to moderate Reynolds numbers. Sellier [6] studied a similar problem in the lubrication limit and even found an analytical solution. We now validate our results with the results in the literature. Instead of a periodic free surface we prescribe a step down/step up and a double bell free surface shape. The results are shown in Figure 5. We observe a perfect agreement of the two solutions. It can be concluded that the model predicts the right behavior in the validity domain of Sellier's approach.

We finally test the finite element solution by comparing the numerical results with an analytical approximation. We therefore assume in (5) that the free surface $h = 1 + A \sin x$ has only a very weak undulation or equivalently $A \ll 1$. This allows us to treat *A* as a perturbation parameter and we can expand the unknown film thickness in a perturbation series which reads $f = 1 + Af_1$. Substituting this ansatz into (5) leads to a linear ordinary differential equation for f_1 which can be solved easily.



a) Step down and step up free surface profile.



b) “Double” bell free surface profile profile.

Figure 5. Comparison of the analytical solution of Sellier [6] and the numerical solution within the WRIBL framework. The parameters for the WRIBL approach are Re=0.1, Bo^{-1}=30, cotα=0, δ=0.1, [7].

Assuming periodic boundary conditions, we arrive at the solution $f_1 = S_1 \sin x + C_1 \cos x$ with two constants S_1 and C_1.

As a measure for the bottom topography we take again the Fourier decomposition and compare the analytical and the numerical results for increasing *A*. Results of the comparison are presented in Figure 6. We study the amplitudes vs. the Reynolds number and the free surface amplitude. Obviously, the agreement for very small *A* is perfect for all Reynolds numbers. Increasing *A* further the agreement is still reasonable for intermediate to larger Reynolds numbers.

We remark that the bottom amplitude shows a local maximum at a certain Reynolds number. A similar effect has been observed for the corresponding forward problem where the free surface amplitude shows a maximum .

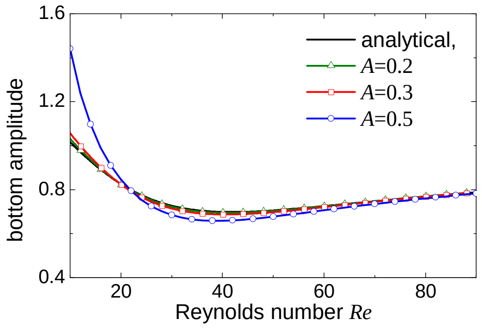


Figure 6. Comparison of the bottom amplitude from the analytic perturbation approximation and the numerical solution. The solid line denotes the analytic solution for $A \ll 1$. Lines with symbols indicate numeric solutions. The fixed parameters are Bo^{-1}=10, cotα=1, δ=0.2, [7].

4. Linear stabiliy analysis

In the previous sections we computed steady solutions for the problem (2) and (3). At first glance unsteady solutions are not reasonable since the bottom topography is a solid wall. However, the unsteady problem arises in a different context when considering the problem as a new direct problem where the free surface is unknown. The approach is the following: First we prescribe a free surface and seek the corresponding bottom topography. In a second step we consider the forward problem for this bottom topography. In this forward problem the steady solution coincides with the steady free surface, nevertheless, the unsteady solution shows a different behavior. Unsteady fluctuations in the form of surface waves are possible.

As a first step to such an unstable configuration we consider the linear stability. That means we prescribe a free surface, compute the topography and find conditions which guarantee stable or unstable free surfaces.

We first linearize (2) and (3) around the steady solution and substitute $q = 1 + \varepsilon \tilde{q}(x,t)$ and $f = f_S(x) + \varepsilon \tilde{f}(x,t)$ with ε small. This yields a linear PDE for $\tilde{q}, \tilde{f}$. We prescribe the perturbations as normal modes and make an ansatz

$$\tilde{f} = \hat{f}(x) e^{i(kx-\omega t)}, \quad \tilde{q} = \tilde{q}(x) e^{i(kx-\omega t)}. \qquad (6)$$

Finally, we obtain a linear eigenvalue problem with eigenvalue ω and spatial wavenumber k which is solved numerically using COMSOL. The sign of the imaginary part of ω gives information about the stability. If $\mathrm{Im}(\omega)>0$ then the free surface is unstable, otherwise it is stable. The numerical solution allows us to determine a stability chart. We expect that for A=0 the system corresponds to the flow over a flat bottom where the critical Reynolds number reads $\mathrm{Re}_{crit}=5/6\cot\alpha$. Figure 7 shows the critical Reynolds number for different inverse Bond numbers and increasing free surface amplitude. We find that in most cases the free surface topography leads to an increase in the critical Reynolds number which corresponds to a stabilization of the system. However, for higher inverse Bond numbers the effect changes and the critical Reynolds number decreases.

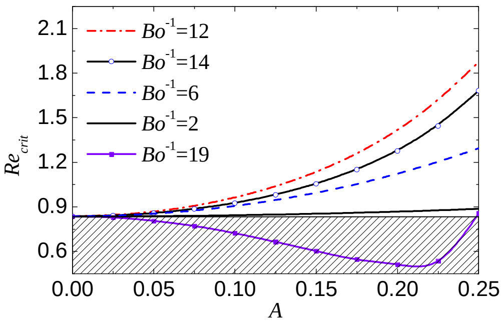


Figure 7. Critical Reynolds number vs. free surface amplitude and different inverse Bond numbers for the case $\cot\alpha=1$, $\delta=0.2$. The dashed domain indicates the critical Reynolds number for the flat bottom $\mathrm{Re}_{crit}=5/6\cot\alpha\approx0.833$, [7].

7. Conclusions

We study the film flow of a viscous liquid flowing down an incline. Instead of the widely studied direct problem of finding the unknown free surface we focus on the corresponding inverse problem. We therefore prescribe a free surface shape and seek the bottom topography. Starting from a weighted-residual integral boundary-layer model we solve for the steady bottom topography using finite elements. We reveal that wavy free surfaces require strongly undulated topographies with steep troughs. Parametric studies show that this effect increases with increasing free surface amplitude.

In order to validate our model and the numerical method we compare our results to analytical results in the literature based on the lubrication approximation. Another comparison is provided by an analytical perturbation approximation for small free surface amplitudes. Both comparisons show that the WRIBL model delivers reasonable results up to moderate Reynolds numbers.

Finally, we study the stability of the steady solution. We conclude that the free surface undulation provides a mechanism to stabilize the free surface. However, for larger surface tension, this effect is decreased and the critical Reynolds number decreases.

8. References

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