Control Synthesis for Distributed Parameter Systems Modeled by FEM in COMSOL Multiphysics

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Abstract: In the paper, first a basic concept of distributed parameter system models in the form of lumped-input/distributed-output systems is introduced. Next, modeling of temperature fields of a melting glass process as distributed parameter system by using of finite element method based software package COMSOL Multiphysics is presented. Distributed characteristics either in stationary or transient form obtained there by numerical solution are exported to the MATLAB & Simulink Environment, where lumped and distributed models for control synthesis purpose are created. Finally, by means of Distributed Parameter Systems Blockset for MATLAB & Simulink - Third-Party Product of The MathWorks, feedback distributed parameter system of control is arranged. There for chosen control synthesis methods, simulation of control process of temperature field of melting glass is executed and analyzed.

Keywords: distributed parameter systems, modeling, control, temperature fields.

1. Introduction

Technological processes in the engineering practice from point of view of systems and control theory are frequently in the form of distributed parameter systems (DPS) with dynamics defined on complex-shape 3D definition domains. Modeling and simulation in this area is now widely accepted as an important tool in product design and process development to improve both productivity and quality. For modeling and dynamical analysis of DPS wide possibilities are offered by COMSOL Multiphysics Environment based on efficient numerical solution of sets of partial differential equations by finite element method (FEM).

Techniques of FEM based modeling and design of control synthesis methods of DPS which is acceptable for various technological processes, is demonstrated on modeling and simulation of control of temperature fields of the glass melting furnace. For control synthesis purpose, numerical models in the form of a lumped-input/distributed-output system (LDS) were created by means of numerical dynamical – FEM solution.

Analysis of formulated models, simulation of temperature fields, but also design of control synthesis was carried out using the Distributed Parameter Systems Blockset for MATLAB & Simulink - Third-party software product of The MathWorks, developed at the Institute of Automation, Measurement and Applied Informatics, Faculty of Mechanical Engineering, Slovak University of Technology [7].

2. LDS models of DPS

Generally, DPS are systems whose state/output variables, $X(x,y,z,t)/Y(x,y,z,t)$ are distributed variables or fields of variables, where (x,y,z) are spatial coordinates in 3D. In the mathematical theory, dynamics of DPS is described by partial differential equations (PDE) with boundary and initial conditions, [1], [2], [3]. As to the input/output relations, PDE define distributed - input/distributed - output systems (DDS) between distributed input, $U(x,y,z,t)$ and distributed output variables, $Y(x,y,z,t)$, at initial and boundary conditions given. Distributed parameter systems very frequently are found in various technical and non-technical branches in the form of LDS, [5], see Fig. 1.

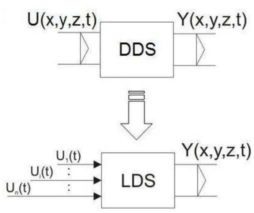


Figure 1. Lumped-input/distributed-output system representation of distributed parameter systems.

2.1 Dynamics of LDS

Output of the linear LDS in time domain, or in s-domain is in the form:

$$Y\left(\overline{x},t\right)=\sum_{i=1}^{n}Y_i\left(\overline{x},t\right)=\sum_{i=1}^{n}\mathcal{G}_i\left(\overline{x},t\right)\otimes U_i\left(t\right) \qquad (1)$$

$$Y\left(\overline{x},s\right)=\sum_{i=1}^{n}Y_i\left(\overline{x},s\right)=\sum_{i=1}^{n}S_i\left(\overline{x},s\right)U_i\left(s\right) \qquad (2)$$

where $\overline{x}=\left(x,y,z\right)$ is position vector in 3D, $U_i\left(t\right)$ - lumped input quantity, $\mathcal{G}_i\left(\overline{x},t\right)$ - i-th distributed impulse response, $\otimes$ - denotes convolution product, $Y_i\left(\overline{x},t\right)$ - system response to the i-th input, $S_i\left(\overline{x},s\right)$ - i-th transfer function, [6]. When $U_i\left(t\right)$ is a unit-step (Heaviside) function, $Y_i\left(\overline{x},t\right)$ is in the form of distributed transient response function $\mathcal{H}_i\left(\overline{x},t\right)$.

For a discrete-time system considering zero-order hold (ZOH) units, the overall distributed output variable of LDS and ZOH can be expressed in the form:

$$Y\left(\overline{x},k\right)=\sum_{i=1}^{n}Y_i\left(\overline{x},k\right)=\sum_{i=1}^{n}\mathcal{G}H_i\left(\overline{x},k\right)\oplus U_i\left(k\right) \qquad (3)$$

where $\oplus$ denotes convolution sum. For points $\overline{x}_i=\left(x_i,y_i,z_i\right)$ located in surroundings of lumped input variables, $U_i\left(t\right)$, where partial distributed transient responses $\mathcal{H}_i\left(\overline{x}_i,t\right)$ attain maximal amplitudes, partial particular distributed output variables are obtained in time-domain, or in z-domain respectively:

$$\left\{Y_i\left(\overline{x}_i,k\right)=\mathcal{G}H_i\left(\overline{x}_i,k\right)\oplus U_i\left(k\right)\right\}_{i=1,n} \qquad (4)$$

$$\left\{Y_i\left(\overline{x}_i,z\right)=SH_i\left(\overline{x}_i,z\right)U_i\left(z\right)\right\}_{i=1,n} \qquad (5)$$

For the space dependency and in steady-state we can define reduced transient step responses between i-th input variable at point $\overline{x}_i=\left(x_i,y_i,z_i\right)$ and corresponding partial particular distributed output variable in steady-state:

$$\left\{\mathcal{H}HR_i\left(\overline{x},\infty\right)=\frac{\mathcal{H}H_i\left(\overline{x},\infty\right)}{\mathcal{H}H_i\left(\overline{x}_i,\infty\right)}\right\}_{i=1,n} \qquad (6)$$

Dynamics of LDS is decomposed to time and space components. In the time dependency, there are, discrete transfer functions of the form $\left\{SH_i\left(\overline{x}_i,z\right)\right\}_{i=1,n}$ and in the space direction there are $\left\{\mathcal{H}HR_i\left(\overline{x},\infty\right)\right\}_{i=1,n}$.

2.2 Feedback control loop based on LDS

Structure of DPS feedback control loop based on LDS is on Fig. 2, where the goal of control is to ensure the steady-state control error to be minimal:

$$\min\left\|\overline{E}\left(x,\infty\right)\right\|=\min\left\|W\left(\overline{x},\infty\right)-Y\left(\overline{x},\infty\right)\right\| \qquad (7)$$

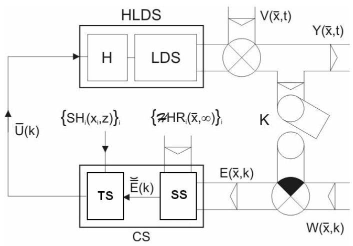


Figure 2. Distributed parameter feedback control loop: HLDS - LDS with zero-order holds $\{H_i\}_i$ on the input, CS - control synthesis, TS - control synthesis in time domain, SS - control synthesis in space domain, $Y\left(\overline{x},t\right)$ - distributed controlled variable, $W\left(\overline{x},k\right)$ - control variable, $V\left(\overline{x},t\right)$ - disturbance variable, $E\left(\overline{x},k\right)$ - control error, K - time/space sampling.

In the block SS, approximation of distributed control error $E\left(\overline{x},k\right)$, on the set of reduced steady-state distributed step responses $\left\{\mathcal{H}HR_i\left(\overline{x},\infty\right)\right\}_i$, is solved.

$$\min_{E_i}\left\|E\left(\overline{x},k\right)-\sum_{i=1}^{n}E_i\left(k\right)\mathcal{H}HR_i\left(\overline{x},\infty\right)\right\|= \\ = \left\|E\left(\overline{x},k\right)-\sum_{i=1}^{n}\breve{E}_i\left(k\right)\mathcal{H}HR_i\left(\overline{x},\infty\right)\right\| \qquad (8)$$

As the solution of approximation problem, the control errors vector $\breve{\overline{E}}\left(k\right)=\left\{\breve{E}_i\left(k\right)\right\}_i$ enters into the block TS, where vector of control variables, $\overline{U}\left(k\right)$, is generated by controllers, $\left\{R_i\left(z\right)\right\}_i$ in single-parameter control loops.

3. Modeling of temperature fields of melting glass in COMSOL Multiphysics

Melting glass process is typical case of DPS. In the input/output relation it is possible to model it as LDS. Lumped inputs set flow rates of heating medium (earth gas and air) into series of burners located on both sides of the glass furnace above molten glass. Temperature field of the molten glass on the definition domain $\Omega \in E_2$ (in cross-section of the melting space) is distributed output variable, see Fig. 3.

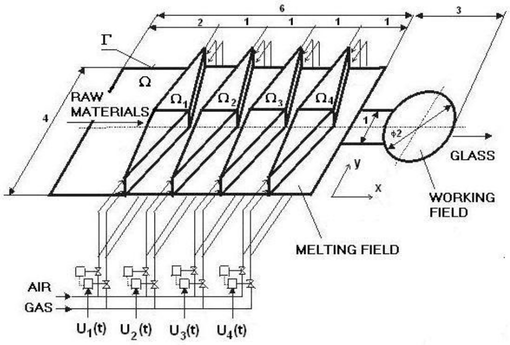


Figure 3. Scheme of recuperative gas glass tank furnace.

Dynamics of a melting glass process as DPS is modeled in COMSOL Multiphysics environment with Heat Transfer Module, which offers very efficient tool for modeling and simulation scientific and engineering problems based on PDE. For the recuperative gas glass tank furnace with the cross flame was made two-dimensional space model of the melting glass dynamics based on FEM. In the definition domain is supposed glass melting in steady-state and heating by the mixture of gas and air, which enter into four pairs of burners. Distribution of temperatures in molten glass in the definition domain $\Omega \in \mathrm{E}_2$ is modeled by PDE of parabolic type

$$\frac{\partial T\left(\bar{x},t\right)}{\partial t} - \nabla\left(aT\nabla\left(\bar{x},t\right)\right) + bT\left(\bar{x},t\right) = \sum_{i=1}^{4} U_i\left(\bar{x},t\right) \qquad (9)$$

with coefficients *a*, *b* and Neumann type boundary conditions for thermal insulation:

$$\vec{n}\left(c\nabla Y\right) + qY = g \qquad (10)$$

where $\vec{n}$ is the outward unit normal and $q = 0$, $g = 0$.

In COMSOL Multiphysics environment, all parameters of PDE and other necessary conditions for simulation and data processing were specified through graphical user interface (GUI). There by means of the toolbar menu in the *Draw* menu 2D geometry model of the melting glass furnace with definition domain $\Omega \in E_2$ and four subdomains Ω_i was drawn.

Next, in the *Physics* menu subdomains of the melting glass with their parameters were defined. Parameters like thermal conductivity, density, heat capacity, absorbtion coefficient, diffusion coeficient and heat source were specified. Also in the Physics menu Boundary conditions were defined.

In the *Mesh* menu mesh for the given geometry was analysed, e.g. initial mesh, various predefined mesh sizes, etc. The Mesh Statistics dialog box, available from the Mesh menu, contains statistical data for the current mesh.

In the *Solve* menu temperature fields of the melting glass either in stationary, or transient form were computed by FEM. Actuating of flow heating medium was performed separately for each pairs of burners on subdomains Ω_i.

Obtained temperature fields were analyzed by means of various forms of plots and their parameters, including AVI animation with help of the *Postprocessing* menu, see Figs. 4 – 6, where are results for heating actuating by pairs of burners in the subdomain Ω_1.

Finally, in the *File* menu Postprocessing data in the form of nodes, elements, data were exported either as txt or m-files to the Matlab - Simulink. There LDS models of DPS for control synthesis purpose are created.

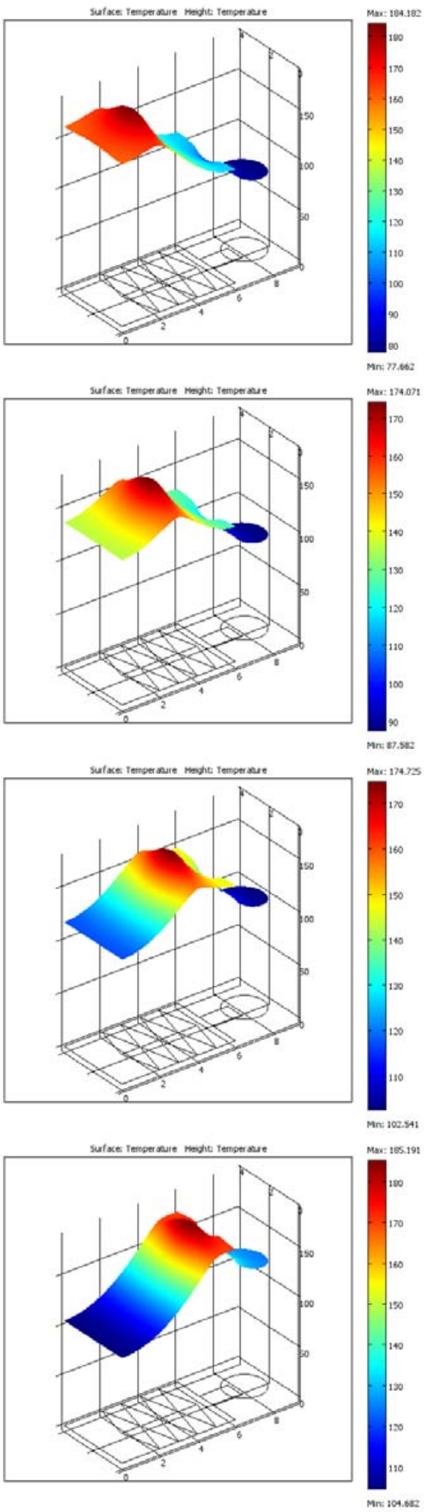


Figure 4. Distributed transient response in steady-state from each lumped input.

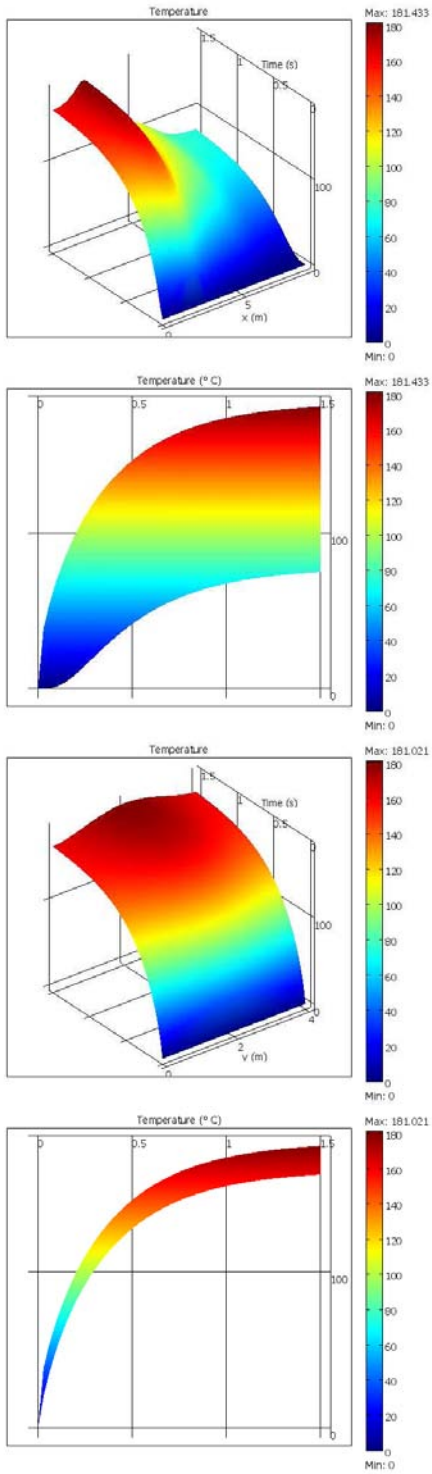


Figure 5. Temperature fields in transient form along *x* and *y* axis actuated by heating in subdomain Ω_1 .

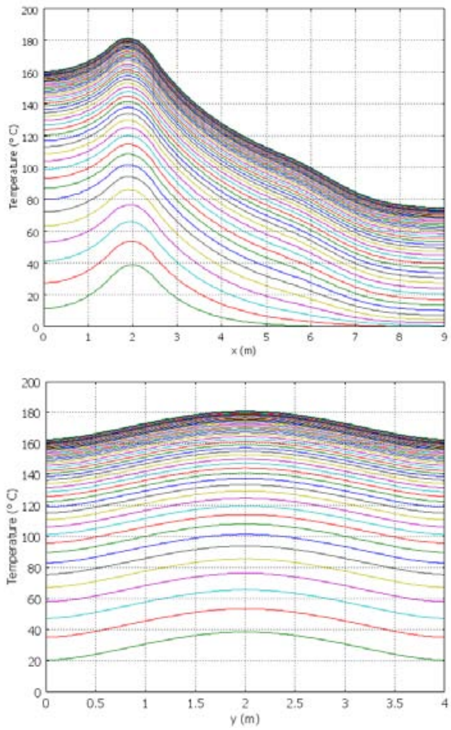


Figure 6. Temperature profiles in time t = [0:300:15000] seconds.

4. DPS Blockset for MATLAB & Simulink

For a software support of modeling, control and design of Distributed Parameter Systems given on complex 3D domains of definition, DPS Blockset for MATLAB & Simulink as Third-Party MathWorks Product has been developed [7], see Fig. 7. The library of DPS Blockset shows Fig. 8.

The block *HLDS* models controlled distributed parameter systems as lumped-input/distributed-output systems with zero-order hold units. The *DPS Control Synthesis* provides feedback to distributed parameter controlled systems in control loops with blocks for PID, algebraic, state space and robust control. The block *DPS Input* generates distributed quantities which can be used as distributed control variables or distributed disturbances, etc. *DPS Display* presents distributed quantities with many options including export to AVI files. The block *DPS Space Synthesis* performs space synthesis as an approximation problem. The block *Tutorial* presents methodological framework for formulation and solution of distributed parameter systems of control. The block *Show* contains motivation examples. The block *Demos* contains examples oriented to methodology of modeling and control synthesis. The *DPS Wizard* in step-by-step operation, by means of five model examples on 1D-3D with default parameters, gives a guide for arrangement and setting distributed parameter control loops.

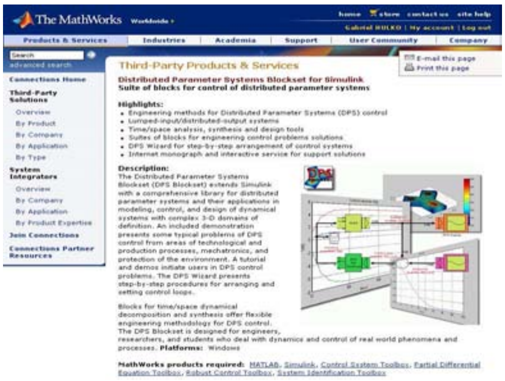


Figure 7. PS Blockset on the web portal of The MathWorks.

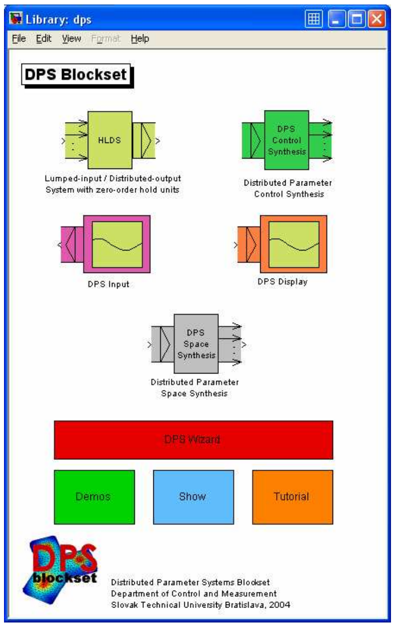


Figure 8. The library of DPS Blockset for MATLAB & Simulink.

5. Synthesis of control and simulation of control process

Temperature fields of the melting glass obtained by FEM solution in COMSOL Multiphysics were exported to Matlab-Simulink, in order to determine lumped and distributed models. Transfer functions $\left\{SH_i\left(\overline{x}_i, z\right)\right\}_{i=1,4}$ are used for control synthesis in time domain TS and reduced transient step responses in steady-state $\left\{\mathcal{H}HR_i\left(\overline{x}, \infty\right)\right\}_{i=1,4}$ serves as basis functions for solution of approximation task in the control synthesis in space domain SS.

Transfer functions were identified in GUI *ident* of the System Identification Toolbox for points $\left\{\overline{x}_i = \left(x_i, y_i\right)\right\}_{i=1,4}$ in each subdomain Ω_i, where temperatures attain maximal values, e.g. for actuating in subdomain Ω_1 result of identification is on Fig. 9 and obtained discrete transfer function is in the form:

$$SH_1\left(\overline{x}_1, z\right) = \frac{38,97z^2 - 47,96z + 13,85}{z^3 - 1,61z^2 + 0,65z - 0,0169} \qquad (11)$$

Simulation of the control process with four PID controllers was realised in the DPS Blockset for MATLAB & Simulink. By means of blocks of the DPS Blockset, DPS feedback control system with four PID controllers embedded in the DPS PID Synthesis block was arranged, see Fig. 10. Synthesis of PID controllers was adjusted in order to assure aperiodic running of the quadratic norm of distributed control error. Control objective is certain temperature required by technology with 10% divergence at most. Results of the PID control process for given reference quantity are on Fig. 11.

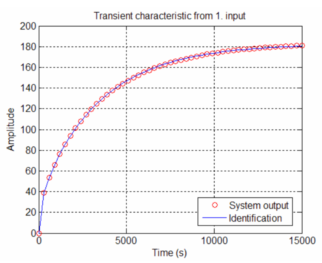


Figure 9. Identification of partial transient response in subdomain Ω_1.

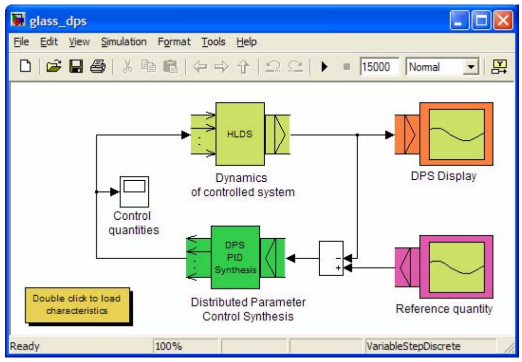
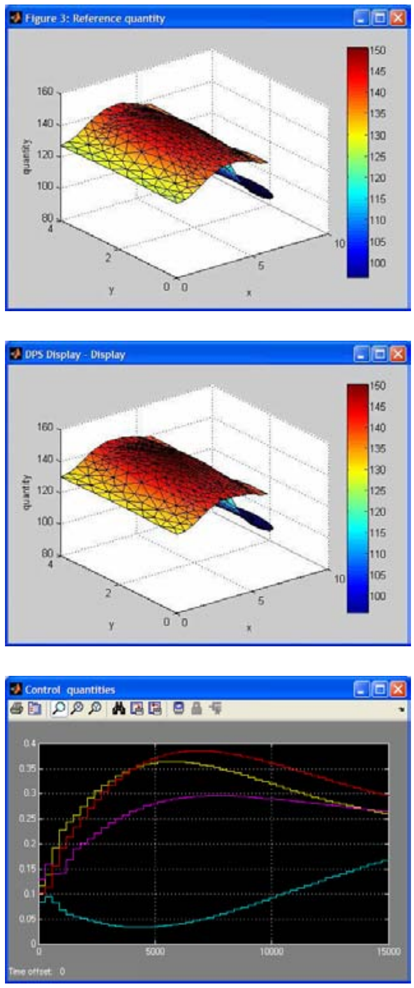


Figure 10. PID feedback control loop in DPS Blockset.



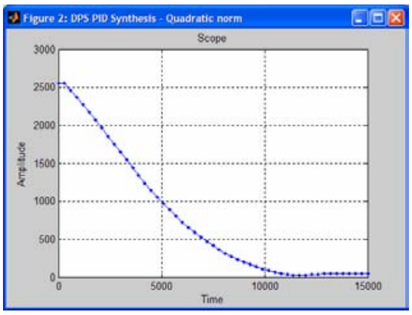


Figure 11. PID control of temperature field of the melting glass: distributed reference quantity $W(\bar{x},\infty)$, controlled quantity $Y(\bar{x},\infty)$, control quantities $U_i(k)$ and quadratic norm of distributed control error $\|E(\bar{x},k)\|$.

6. Conclusions

Nowadays, a number of software products, environments and virtual try-out spaces for numerical dynamical analysis of machines and processes are at disposal, practically in all engineering disciplines. Based on powerful FEM solvers and advanced specific options, COMSOL Multiphysics provides an efficient and accurate solution which users need.

Methodical approach presented in the paper demonstrates simple possibilities, how to exploit of distributed dynamical characteristics, obtained by numerical analysis of systems on complex definition domains for design of various control synthesis methods for distributed parameter systems. The DPS Blockset for MATLAB & Simulink provides block-oriented efficient software for this kind of tasks.

7. References

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